

multipole expansion

$$\frac{1}{|\vec{r} - \vec{r}'|} = \frac{1}{\sqrt{r^2 - 2\vec{r} \cdot \vec{r}' + r'^2}} = \sum_{k=0}^{\infty} \frac{r_{<}^k}{r_{>}^{k+1}} P_k(\hat{r} \cdot \hat{r}')$$

using generating function of Legendre polynomials

$$\frac{1}{\sqrt{1 - 2xt + t^2}} = \sum_{n=0}^{\infty} P_n(x) t^n$$

$$P_n(x) = \frac{1}{n!} \left. \frac{d^n}{dt^n} \right|_{t=0} \frac{1}{\sqrt{1 - 2xt + t^2}}$$

$$P_0(x) = 1, \quad P_1(x) = x$$

take derivative of generating function wrt t and compare powers of t

$$(n + 1)P_{n+1}(x) = (2n + 1)xP_n(x) - nP_{n-1}(x)$$

addition theorem for spherical harmonics

$$P_l(\hat{r} \cdot \hat{r}') = \frac{4\pi}{2l+1} \sum_{m=-l}^l \overline{Y_{lm}(\hat{r})} Y_{lm}(\hat{r}')$$

expansion of Coulomb repulsion in terms of spherical harmonics

$$\frac{1}{|\vec{r} - \vec{r}'|} = \sum_{k=0}^{\infty} \frac{r_{<}^k}{r_{>}^{k+1}} \frac{4\pi}{2k+1} \sum_{m=-k}^k \overline{Y_{km}(\hat{r})} Y_{km}(\hat{r}')$$

Coulomb matrix elements

$$\Phi_a(\vec{r}) = \frac{u_{n_a l_a}(r)}{r} Y_{l_a m_a}(\vartheta, \varphi)$$

$$\int d^3 r \int d^3 r' \overline{\Phi_4(\vec{r}) \Phi_3(\vec{r}')} \frac{1}{|\vec{r} - \vec{r}'|} \Phi_2(\vec{r}') \Phi_1(\vec{r})$$

$$= \sum_{k=0}^{\infty} \int dr \overline{u_4(r)} u_1(r) \int dr' \overline{u_3(r')} u_2(r') \frac{r_{<}^k}{r_{>}^{k+1}}$$

$$\times \frac{4\pi}{2k+1} \sum_{\mu=-k}^k \int d\hat{r} \overline{Y_4(\hat{r}) Y_{k\mu}(\hat{r}) Y_1(\hat{r})} \int d\hat{r}' \overline{Y_3(\hat{r}') Y_{k\mu}(\hat{r}') Y_1(\hat{r}')}$$