multipole expansion

$$\frac{1}{|\vec{r}-\vec{r}\,'|} = \frac{1}{\sqrt{r^2 - 2\vec{r}\cdot\vec{r}\,' + r'^2}} = \sum_{k=0}^{\infty} \frac{r<^k}{r_>^{k+1}} P_k(\hat{r}\cdot\hat{r}\,')$$

using generating function of Legendre polynomials

$$\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x)\, t^n$$

$$P_n(x) = \frac{1}{n!}\,\frac{d^n}{dt^n}\bigg|_{t=0} \frac{1}{\sqrt{1-2xt+t^2}}$$

$$P_0(x) = 1\,,\;\; P_1(x) = x$$

take derivative of generating function wrt *t* and compare powers of *t*

$$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x)$$

addition theorem for spherical harmonics

$$P_l(\hat{r}\cdot\hat{r}') = \frac{4\pi}{2l+1}\sum_{m=-l}^{l}\overline{Y_{lm}(\hat{r})}\,Y_{lm}(\hat{r}\,')$$

expansion of Coulomb repulsion in terms of spherical harmonics

$$\frac{1}{|\vec{r}-\vec{r}\,'|} = \sum_{k=0}^{\infty}\frac{r_<^k}{r_>^{k+1}}\,\frac{4\pi}{2k+1}\sum_{m=-k}^{k}\overline{Y_{km}(\hat{r})}\,Y_{km}(\hat{r}\,')$$

Coulomb matrix elements

$$\Phi_a(\vec{r}) = \frac{u_{n_a l_a}(r)}{r} \, Y_{l_a m_a}(\vartheta, \varphi)$$

$$\int d^3 r \int d^3 r' \, \overline{\Phi_4(\vec{r})} \, \overline{\Phi_3(\vec{r}')} \, \frac{1}{|\vec{r} - \vec{r}'|} \Phi_2(\vec{r}') \, \Phi_1(\vec{r})$$

$$= \sum_{k=0}^{\infty} \int dr \, \overline{u_4(r)} \, u_1(r) \int dr' \, \overline{u_3(r')} \, u_2(r') \frac{r_<^k}{r_>^{k+1}}$$

$$\times \frac{4\pi}{2k+1} \sum_{\mu=-k}^{k} \int d\hat{r} \, \overline{Y_4(\hat{r})} \, \overline{Y_{k\mu}(\hat{r})} \, Y_1(\hat{r}) \int d\hat{r}' \, \overline{Y_3(\hat{r}')} \, Y_{k\mu}(\hat{r}') \, Y_1(\hat{r}')$$