exactly solvable problems

It might be noted here, for the benefit of those interested in exact solutions, that there is an alternative formulation of the **many-body problem**, i.e., *how many bodies are required before we have a problem?*

G.E. Brown points out that this can be answered by a look at history.

- In eighteenth-century Newtonian mechanics, the **three**-body problem was insoluble.
- With the birth of general relativity around 1910 and quantum electrodynamics in 1930, the **two**- and **one**-body problems became insoluble.
- And within modern quantum field theory, the problem of **zero** bodies (vacuum) is insoluble.

So, if we are out after exact solutions, no bodies at all is already too many!

R.D. Mattuck: A Guide to Feynman Diagrams in the Many-Body Problem

Rayleigh-Schrödinger perturbation theory

$$\hat{H} = \hat{H}_0 + \lambda \hat{H}_1$$

perturbation of non-degenerate eigenstate $\hat{H}_0 |n^{(0)}\rangle = E_n^{(0)} |n^{(0)}\rangle$

ansatz (not normalized!)

$$E_n = E_n^{(0)} + \lambda E_n^{(1)} + \lambda^2 E_n^{(2)} + \lambda^3 E_n^{(3)} + \cdots$$

$$|n\rangle = |n^{(0)}\rangle + \lambda |n^{(1)}\rangle + \lambda^2 |n^{(2)}\rangle + \lambda^3 |n^{(3)}\rangle + \cdots$$

solve power-by-power

$$E_n^{(0)} = \langle n^{(0)} | \hat{H}_0 | n^{(0)} \rangle$$

$$E_n^{(1)} = \langle n^{(0)} | \hat{H}_1 | n^{(0)} \rangle \qquad |n^{(1)}\rangle = \sum_{m \neq n} |m^{(0)}\rangle \frac{\langle m^{(0)} | \hat{H}_1 | n^{(0)} \rangle}{E_n^{(0)} - E_m^{(0)}}$$

$$E_n^{(2)} = \langle n^{(0)} | \hat{H}_1 | n^{(1)} \rangle = \sum_{m \neq n} \frac{|\langle m^{(0)} | \hat{H}_1 | n^{(0)} \rangle|^2}{E_n^{(0)} - E_m^{(0)}}$$

Rayleigh-Schrödinger perturbation theory

perturbation of degenerate state $\hat{H}_0 |n_\alpha^{(0)}\rangle = E_n |n_\alpha^{(0)}\rangle$

$$|\tilde{n}_\alpha^{(0)}\rangle = \sum_s c_{\alpha\beta} |n_\beta^{(0)}\rangle$$

$$\left(\hat{H}_0 - E_n^{(0)}\right) |\tilde{n}_\beta^{(1)}\rangle = \left(E_{n,\alpha}^{(1)} - \hat{H}_1\right) |\tilde{n}_\alpha^{(0)}\rangle$$

eigenvalue problem for $E^{(1)}$

$$\begin{pmatrix} \langle n_1^{(0)}|\hat{H}_1|n_1^{(0)}\rangle & \langle n_1^{(0)}|\hat{H}_1|n_2^{(0)}\rangle & \cdots & \langle n_1^{(0)}|\hat{H}_1|n_d^{(0)}\rangle \\ \langle n_2^{(0)}|\hat{H}_1|n_1^{(0)}\rangle & \langle n_2^{(0)}|\hat{H}_1|n_2^{(0)}\rangle & \cdots & \langle n_2^{(0)}|\hat{H}_1|n_d^{(0)}\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle n_d^{(0)}|\hat{H}_1|n_1^{(0)}\rangle & \langle n_d^{(0)}|\hat{H}_1|n_2^{(0)}\rangle & \cdots & \langle n_d^{(0)}|\hat{H}_1|n_d^{(0)}\rangle \end{pmatrix} \begin{pmatrix} c_{\alpha,1} \\ c_{\alpha,2} \\ \vdots \\ c_{\alpha,s} \end{pmatrix} = E_{n,\alpha}^{(1)} \begin{pmatrix} c_{\alpha,1} \\ c_{\alpha,2} \\ \vdots \\ c_{\alpha,s} \end{pmatrix}$$