Final Exam: Applied Quantum Mechanics WS 2024/25

Duration: 9:30 – 12:00.
No eating/drinking in the lecture room
You may consult your handwritten notes on a single A4 sheet of paper.
No other means (books, calculator, computer, cell phone, etc.) are allowed.
Use permanent ink for writing.
Put your name and student-ID on every sheet of paper on your desk.
Only hand in those pages that you want to be graded.
There are 5 problems for a total of 100 points.

1. particle in a box (20 points)

   (a) Consider a potential well with potential $V(x) = 0$ for $0 < x < L$ and infinite otherwise. What are the eigenenergies E_n and the corresponding normalized eigenfunctions $\varphi_n(x)$? What values can the quantum number n have?

   (b) Now consider a two-dimensional potential well with $V(x, y){=}0$ for $0{<}x{<}L_x$ and $0{<}y{<}L_y$, and infinite otherwise. What are the normalized eigenfunctions and eingenenergies?

   (c) For the special case $L_x{=}2L_y$ find the lowest *degenerate* energy level and write down the quantum numbers of all orthogonal eigenfunctions with that energy.

2. potential step (20 points)
   Consider a potential step

$$V(x) = \begin{cases} 0 & \text{for } x < 0 \\ V_0 & \text{for } x > 0 \end{cases}$$

   with $V_0 \leq 0$. An electron of energy $E > 0$ is incident from the left.

   (a) Make a sketch of the potential.

   (b) Find the solution of the Schrödinger equation by matching at $x = 0$.

   (c) Determine the probability current density for $x < 0$ and for $x > 0$.

   (d) Calculate the probabilities $R(E)$ for the electron to be reflected and $T(E)$ for it to be transmitted as a function of its energy E.

3. linear potential (20 points)

   (a) The Airy functions solve the differential equation $f''(z) = zf(z)$. Show that

   $$\varphi(x) = f\left(\left(2m_e\alpha/\hbar^2\right)^{1/3}(x - E/\alpha)\right)$$

   solves the Schrödinger equation

   $$-\frac{\hbar^2}{2m_e}\frac{d^2\varphi(x)}{dx^2} + \alpha x\,\varphi(x) = E\varphi(x)$$

   (b) Now consider a piece-wise potential $V(x)$ infinite for $x<0$ and $V(x) = \alpha x$ (with $\alpha>0$) for $x>0$.

      i. Sketch $V(x)$ and the lowest three eigenfunctions, paying attention to the boundary conditions and the number of nodes.

      ii. Find the eigenfunctions and their eigenenergies in terms of the Airy functions Ai, Bi and their roots a_n, b_n (i.e. $\mathrm{Ai}(a_n) = 0 = \mathrm{Bi}(b_n)$).
      Are the eigenenergies positive or negative?

4. harmonic oscillator in a weak electric field (20 points)
   Consider a one-dimensional harmonic oscillator with Hamiltonian

   $$H_0 = -\frac{1}{2}\frac{d^2}{dx^2} + \frac{x^2}{2} = a^\dagger a + \frac{1}{2} \qquad \text{where } a = \frac{1}{\sqrt{2}}\left(x - \frac{d}{dx}\right)$$

   with eigenenergies $E_n = n+\frac{1}{2}$ and normalized eigenstates $|n\rangle = a^\dagger|n-1\rangle/\sqrt{n}$, $a|0\rangle = 0$.
   It is perturbed by a weak electric field $H_1 = \alpha x$.

   (a) Show that the matrix elements of the unperturbed harmonic oscillator states with the perturbation are

   $$\langle m|H_1|n\rangle = \alpha\left(\sqrt{n}\,\delta_{m,n-1} + \sqrt{n+1}\,\delta_{m,n+1}\right)/\sqrt{2}$$

   (b) Using these matrix elements, calculate the change in energy in first and second order of perturbation theory.

5. hydrogen atom (20 points)
   Consider the hydrogen $3p$ states

   $$\varphi_{n=3,l=1,m}(\vec{r}) = \frac{u_{3,1}(r)}{r}Y_{1,m}(\vartheta,\varphi)$$

   (a) What is the energy of these states?
   List the quantum numbers of all hydrogen states having the same energy?
   What is the degeneracy of the energy level?

   (b) Sketch the radial function $u_{3,1}(r)$. How many nodes does it have?
   Indicate in your sketch how it behaves for r close to the origin and for very large r.

   (c) Give all the points (surfaces) where $\varphi_{3,1,0}(\vec{r})$ vanishes.