

Exercise Sheet 8 due 13 December 2018

1. Legendre polynomials

The Legendre polynomials $P_n(x)$ are defined by the recursion relation

$$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x) \quad (1)$$

with $P_0(x) = 1$ and $P_1(x) = x$.

- i. Write a program to calculate a symbolic representation of the Legendre polynomials, plot the $P_n(x)$ on the range $x \in [-1, 1]$, and verify that the P_n are orthogonal on the interval $[-1, 1]$: $\int_{-1}^1 P_n(x) P_m(x) dx = \frac{2}{2n+1} \delta_{n,m}$.
- ii. Show (by induction) that the Legendre polynomials are the solutions to the eigenvalue problem

$$-\left(\frac{d}{dx}(1-x^2)\frac{d}{dx}\right)P_n(x) = n(n+1)P_n(x). \quad (2)$$

For this you will also need to show (by induction) that

$$(1-x^2)\frac{dP_n(x)}{dx} = n(P_{n-1}(x) - xP_n(x)). \quad (3)$$

Show that with the substitution $x = \cos(\theta)$ this becomes the eigenvalue equation for \vec{L}^2 for $m = 0$. Write the spherical harmonics $Y_{l,m=0}(\theta, \phi)$ in terms of the Legendre polynomials.

2. spin

Compare the commutators of the Pauli matrices of last week's exercises with the commutation relation of the components of the orbital angular momentum \hat{L}_x , \hat{L}_y , and \hat{L}_z . Relate the eigenvalues of $\vec{\sigma}^2$ and $\hat{\sigma}_z$ to what we derived for general angular-momentum operators.

3. Angular momentum operator in spherical coordinates

Given

$$\begin{aligned} \hat{L}_x &= i\hbar \left(+\sin\phi \frac{\partial}{\partial\theta} + \cot\theta \cos\phi \frac{\partial}{\partial\phi} \right) \\ \hat{L}_y &= i\hbar \left(-\cos\phi \frac{\partial}{\partial\theta} + \cot\theta \sin\phi \frac{\partial}{\partial\phi} \right) \\ \hat{L}_z &= -i\hbar \frac{\partial}{\partial\phi} \end{aligned}$$

- i. Show that $\hat{L}_{\pm} = \hat{L}_x \pm i\hat{L}_y = \hbar e^{\pm i\phi} \left(\pm \frac{\partial}{\partial\theta} + i \cot\theta \frac{\partial}{\partial\phi} \right)$
- ii. Show that $\hat{L}^2 = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2 = -\hbar^2 \left(\frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial}{\partial\theta} \right) + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\phi^2} \right)$
Hint: $\hat{L}^2 = \hat{L}_+ \hat{L}_- + \hat{L}_z(\hat{L}_z - \hbar)$