Exercise Sheet 7 due 28 November

1. *Hermitian matrices*

   i. Show, for Hermitian operators $\hat{A}$ and $\hat{B}$, that the product $\hat{A}\hat{B}$ is Hermitian if and only if $\hat{A}$ and $\hat{B}$ commute.

   ii. Prove that the operator that is the commutator $[\hat{A}, \hat{B}]$ of two Hermitian operators $\hat{A}$ and $\hat{B}$ is never Hermitian, unless it is zero. Do you see how to modify the non-vanishing commutator so that it becomes Hermitian?

2. *Pauli matrices*

The Pauli matrices are defined as

$$\hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \hat{\sigma}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

   i. Calculate $\vec{\sigma}^2 = \hat{\sigma}_x^2 + \hat{\sigma}_y^2 + \hat{\sigma}_z^2$.

   ii. Find the eigenvalues and (normalized) eigenvectors $|\chi_{z,n}\rangle$ of $\hat{\sigma}_z$.

   iii. Find the eigenvalues and (normalized) eigenvectors $|\chi_{x,n}\rangle$ of $\hat{\sigma}_x$.

   iv. Show by explicit calculation that $\sum_n |\chi_{x,n}\rangle\langle\chi_{x,n}|$ is the identity matrix.

   v. Determine the commutators between each pair of the Pauli matrices by explicit matrix multiplication. Simplify your answer as much as possible and write the result in terms of the Pauli matrices.

   vi. Calculate $\exp(\sigma_x)$ by transforming to the eigenbasis and, alternatively, by using the power series.