

Exercise Sheet 6 due 21 November

1. operators and expectation values

- i. If the eigenenergies of the Hamiltonian \hat{H} are E_n and the eigenfunctions are φ_n , what are the eigenvalues and eigenfunctions of the operator $\hat{H}^2 - \hat{H}$?
- ii. Consider the equal linear superposition $|\varphi\rangle = (|\varphi_1\rangle + |\varphi_2\rangle)/\sqrt{2}$ of the two lowest eigenstates of an infinitely deep potential well. What are the expectation values of
 - (a) the energy $(\int \overline{\varphi(x)} \hat{H} \varphi(x) dx)$,
 - (b) the momentum $(\int \overline{\varphi(x)} \hat{p} \varphi(x) dx)$, and
 - (c) the position $(\int \overline{\varphi(x)} x \varphi(x) dx)$?

2. Hermitian and Unitary operators

- i. Show that the momentum operator $\hat{p} = -i\hbar\vec{\nabla}$ is Hermitian. For simplicity, you may perform this proof for a one-dimensional system. (Why?)

Hint: Consider $\int \overline{\varphi_n(x)} \hat{p}_x \varphi_m(x) dx$, where the $\varphi_n(x)$ are a complete orthonormal set and integrate by parts. Note that the $\varphi_n(x)$ must vanish at infinity, as otherwise they could not be normalized.

- ii. Is the second derivative $\frac{d^2}{dx^2}$ Hermitian?
- iii. Is a Hamiltonian with a real potential Hermitian?
- iv. Show that $\exp(i\hat{M})$ is unitary if \hat{M} is Hermitian.

3. Time-dependence of expectation values

Consider an operator \hat{A} that does not depend on time (i.e., $\partial\hat{A}/\partial t = 0$) and that commutes with the Hamiltonian \hat{H} . Show that the expectation value of this operator, for any state $|\Psi(t, \vec{r})\rangle$, does not depend on time, i.e., $\partial\langle\hat{A}\rangle/\partial t = 0$.

4. Spectral representation

A Hermitian operator \hat{A} has a complete orthonormal set of eigenfunctions $|\varphi_n\rangle$ with associated eigenvalues α_n . Show that we can always write $\hat{A} = \sum \alpha_n |\varphi_n\rangle\langle\varphi_n|$. This expansion in its eigenfunctions is particularly useful for evaluating functions of \hat{A} . To see this, find a simple expression for the inverse, \hat{A}^{-1} . Next, find the spectral representation of $f(\hat{A})$ for some function f .