Exercise Sheet 5 due 22 November 2018

1. *harmonic oscillation*

Consider a harmonic oscillator with Hamiltonian

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}+\frac{m\omega^2}{2}\,x^2$$

and eigenfunctions $H\phi_n = \hbar\omega(n+1/2)\,\phi_n(x)$.
For a wave packet at time $t=0$ given by

$$\Psi(x,t=0)=\sum_n c_n\,\phi_n(x)$$

show that the expectation value of the position of the electron

$$x(t)=\int dx\,\overline{\Psi(x,t)}\,x\,\Psi(x,t)$$

oscillates harmonically with frequency ω: $x(t)=x_0\,\cos(\omega t+\delta)$.
Express x_0 and δ in terms of the amplitudes c_n at $t=0$.
Hint: Use the result of 2(iv) below.

2. *ladder operators*

Consider the harmonic Hamiltonian $H=a^\dagger a\;+1/2$ with operators $a=\left(\zeta+\frac{d}{d\zeta}\right)/\sqrt{2}$ and $a^\dagger=\left(\zeta-\frac{d}{d\zeta}\right)/\sqrt{2}$.

i. Use integration by parts (twice) to show that

$$\int d\zeta\,\overline{\phi_n(\zeta)}\,(H\,\phi_m(\zeta))=\int d\zeta\,(\overline{H\,\phi_n(\zeta)})\,\phi_m(\zeta)$$

ii. Show that the eigenfunctions with different eigenenergies are orthogonal.

iii. Show that $\zeta=(a\;+a^\dagger)/\sqrt{2}$.

iv. Show that for the eigenfunctions $H\,\phi_n=(n+1/2)\,\phi_n$ holds

$$\int d\zeta\,\overline{\phi_n(\zeta)}\,\zeta\,\phi_m(\zeta)=\sqrt{\frac{n+1}{2}}\,\delta_{n,m-1}+\sqrt{\frac{n}{2}}\,\delta_{n,m+1}$$

3. *Advanced: Hermite polynomials*

You can represent a polynomial $a_0+a_1x+a_2x^2+\cdots+a_{n-1}x^{n-1}$ of order smaller than n by an n-dimensional array `a[i]`. Write a code that calculates the lowest 20 Hermite polynomials $H_n(x)$ from the recursion relation

$$H_{n+1}(x)=2x\,H_n(x)-2n\,H_{n-1}(x)$$

starting from $H_0(x)=1$ and $H_1(x)=2x$.

Your code should be able to print the polynomials in symbolic form and to evaluate them numerically so you can produce a plot.