Exercise Sheet 2 due 25 October 2018

1. *probability current density*

   i. Consider a plane wave $\Psi(\vec{r},t) = e^{i(\vec{k}\cdot\vec{r}-\omega(\vec{k})t)}$. What is the corresponding probability current density? Verify the continuity equation.

   ii. *advanced:* Check the continuity equation for a wave packet of the form
   $$\Psi(\vec{r},t) = \int d^3k\, \tilde{\varphi}(\vec{k})\, e^{i(\vec{k}\vec{r}-\omega(\vec{k})t)}$$

2. *time evolution for a particle in a box*

Consider an electron in an infinite potential well of width L. Suppose that the wave function of the electron at time t=0 is $\Psi(x, t = 0) = A \sin^3(\pi x/L)$.

   i. Determine A so that $\Psi(x, t = 0)$ is normalized.
   (Hint: use $\sin(x)=(e^{ix}-e^{-ix})/2i$ to obtain $\sin^2(x)=(1-\cos(2x))/2$ and take the cube; do odd powers of the cos contribute to the integral? rewrite the even power as above)

   ii. Write down the wave function for arbitrary time t and show that it is normalized for any t. (Hint: $4\sin^3(x) = 3\sin(x) - \sin(3x)$)

   iii. Calculate the probability density $|\Psi(x, t)|^2$ as a function of time and plot it. For what times Δt is $|\Psi(x, t + \Delta t)|^2 = |\Psi(x, t)|^2$? What frequency does that correspond to? Does it change for superpositions of more states?

3. *wave packets*

   i. Assume a particle is described by a Gaussian wave packet of width $\sigma_x = 2$ Å at $t = 0$. After what time has the width doubled, given that the particle is (i) an electron, (ii) a proton, (iii) the nucleus of a uranium atom. Now consider a dust particle of mass 1 μg described by a wave packet of width 1 μm. How many years does it take for the width to increase by 1 nm?

   ii. Consider a Gaussian wave packet in a one-dimensional constant potential $V(x) = 0$. Plot the corresponding probability density at different times, and measure its velocity and broadening.

   iii. Now consider the time evolution of the superposition of two Gaussian wave packets with different velocities. Plot the probability density for times before, during, and after the packet have passed each other. (Note: the wave function is describing a *single* electron, so this is *not* describing two electrons passing each other!)

   iv. *advanced:* Can you write down a Gaussian wave packet in a potential $V(x) = 0$ for $x < 0$ with an infinitely high wall at $x = 0$, i.e., $V(x) = \infty$ for $x \geq 0$? Make plots of how the wave packet is reflected at the wall.