1 Particle-Hole Symmetries in Condensed Matter

Martin R. Zirnbauer
Institute of Theoretical Physics
University of Cologne
Zülpicher Str. 77a, 50937 Köln

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E. Pavarini and E. Koch (eds.)
Correlations and Phase Transitions
Modeling and Simulation Vol. 14
Forschungszentrum Jülich, 2024, ISBN 978-3-95806-751-6
http://www.cond-mat.de/events/correl24

1 Introduction

Symmetries have played a major role in the development of physics from ancient history to modern times – be it as exact symmetries leading to exact conservation laws by Noether's theorem and its variants, or as approximate symmetries providing a guide to the essential features of a physical system. In the twentieth century, a major physics strand was the development of the Landau theory (or Landau-Wilson-Ginzburg paradigm) of phase transitions. The hypothesis there was that thermodynamic phases and phase transitions between them can be characterized by the appearance (or disappearance) of a so-called order parameter breaking a continuous or discrete symmetry. Now in recent decades that Landau paradigm was challenged by the discovery of a variety of *topological states* of matter and quantum phase transitions, where different phases are not distinguished by symmetry breaking but rather by some kind of topological invariant taking different values in different phases. Intriguingly, this challenge to the Landau framework has in turn been counter-challenged in recent years: it is currently being investigated [1] whether topological (non-Landau) phase transitions could still be subsumed under a generalized Landau framework invoking a generalized symmetry concept called *higher-form symmetries*, where the conserved "charges" may be magnetic fluxes and or higher-dimensional analogs thereof. Motivated by that general setting, the present lecture focuses on just one detail out of the grand picture: *particle-hole symmetry* as one of the anti-unitary symmetries that play a role in the classification of so-called symmetry-protected topological phases.

In developing the concept of symmetries and their consequences, we are somewhat hampered by the fact that there exists no consensus as to what exactly is meant by a "symmetry" in quantum mechanics. Therefore, our lecture begins with an attempt to offer various definitions and clarifications. Quantum symmetries in our sense act primarily on quantum states, and by Wigner's Theorem they lift to unitary or anti-unitary operations on the Fock-Hilbert space of the quantum system. Since the energy of a stable system (quantum or not) is bounded from below by the ground-state energy (which we may take to be zero), it follows that any operation of symmetry must transform positive-energy eigenstates into other positive-energy eigenstates. So, operations that anticommute with the Hamiltonian (and hence reverses the sign of the energy) cannot qualify as symmetries. Examples of such non-symmetries are chiral "symmetry" or sublattice "symmetry", and Bogoliubov-deGennes "symmetry". We also emphasize that there exists no such thing as a gauge "symmetry"; in fact, gauge invariance can never be broken, neither explicitly nor spontaneously, whereas symmetries can be.

The cacophony of language is particularly severe in the case of particle-hole symmetries. Major review articles say that *every superconductor is particle-hole symmetric*. We strongly discourage the use of such language, as it confuses the notion of symmetry with a structural element of the theoretical formalism. In a similar vein, particle-hole symmetries (usually of antilinear type) must not be confused with the linear symmetry of charge conjugation, which plays a prominent role in the relativistic quantum theory of the Dirac field (as part of the CPT theorem). Understood in our sense, the transformation of a particle-hole symmetry exchanges particle-like excitations of a Fermi-liquid state with hole-like excitations. Acting on a single band, it transforms the Fock vacuum into the totally occupied state. As such it can be a symmetry of the ground state only in a situation of half filling. An informative comparison is with the operation of time-reversal symmetry T: while T acts on (charge density, current density, electric field, magnetic field) as $(\rho, j; E, B) \mapsto (\rho, -j; E, -B)$, a particle-hole symmetry acts as $(\rho, j; E, B) \mapsto (-\rho, j; -E, B)$. Complementary to the anti-unitary symmetry of time reversal, particle-hole symmetries are phenomenologically relevant for systems in zero electric field but nonzero magnetic field.

The present lecture is an excerpt from a review article [2] written a few years ago. Following our basic definitions and list of non-symmetries, we will illustrate the notion of particle-hole symmetry at the example of the Su-Schrieffer-Heeger model and the Kitaev-Majorana chain. Then, simplifying some of the mathematical abstractions in [2], we define particle-hole symmetry *invariantly*, i.e., without fixing any preferred single-particle basis of Hilbert space. We finish the lecture with the fascinating story of composite fermions in the half-filled lowest Landau level.

2 What's a symmetry in quantum mechanics?

In the general setting of quantum theory (and other theories, for that matter) one has two basic structures: *observables* and *states*. These are dual to each other in that there is a *pairing* between them, viz. the operation of taking the expectation value of an observable in a state,

$$\text{observables} \otimes \text{states} \to \text{real numbers}. \tag{1}$$

Physical observables are realized as self-adjoint operators $A = A^\dagger$ on a Hilbert space V. The same goes for states, by what is known as the state's density matrix, ρ, a positive self-adjoint operator of unit trace:

$$\rho = \rho^\dagger, \quad 0 \leq \rho \leq 1, \quad \mathrm{Tr}_V\, \rho = 1. \tag{2}$$

Adopting the Hilbert-space realization, one writes the pairing (1) as a trace:

$$A \otimes \rho \mapsto \mathrm{Tr}_V\, \rho A. \tag{3}$$

The (square of the) overlap between two states (sometimes called the "transition probability" for short) is

$$W_{f \leftarrow i} = \mathrm{Tr}_V\ \rho_f \rho_i\, . \tag{4}$$

The observables $A, B, \ldots$ of quantum theory form an associative algebra: $(AB)C = A(BC)$. For present purposes, note that an *automorphism* of the operator algebra is a transformation $A \mapsto S(A)$ that preserves the operator product,

$$S(AB) = S(A)S(B). \tag{5}$$

Since the algebra of quantum observables is an algebra over $\mathbb{C}$ (the complex numbers) one distinguishes between two kinds of automorphism:

$$S \text{ linear}: \quad S(\lambda A) = \lambda S(A) \quad (\lambda \in \mathbb{C}), \tag{6}$$

$$S \text{ antilinear}: \quad S(\lambda A) = \bar{\lambda} S(A). \tag{7}$$

Remark. The bar operation $\lambda \to \bar{\lambda}$ denotes complex conjugation, i.e. the operation fixing the real axis and inverting the imaginary axis:

$$(\mathrm{Re}\,\lambda, \mathrm{Im}\,\lambda) \mapsto (\mathrm{Re}\,\lambda, -\mathrm{Im}\,\lambda). \tag{8}$$

It should be stressed that (8) is just one out of many automorphisms of $\mathbb{C}$, and from the algebraic viewpoint there is nothing special about the real axis. Indeed, reflection at any line through zero in $\mathbb{C}$ does the same job. If there is only one copy of $\mathbb{C}$ in play, one will simply define the real axis to be that special line of reflection! This freedom of simplification no longer exists for a complex vector space $V \cong \mathbb{C}^n$ of dimension $n > 1$, as one can make different choices of "real axis" in the different subspaces of V. This prompts a warning: in the absence of further structure, no *a priori* notion of complex conjugation exists for a complex vector space V with $\dim V > 1$.

Now, what do we mean by a *symmetry* in quantum mechanics? There are two sets of requirements. The first set is this:

- A symmetry operation, S, is a transformation of the space of states, $0 \le \rho \mapsto S(\rho) \ge 0$.
- S leaves all transition probabilities invariant:

$$\mathrm{Tr}_V\, S(\rho_f)S(\rho_i) = \mathrm{Tr}_V\, \rho_f \rho_i\,. \tag{9}$$

Note that we have been careful to introduce symmetry operations S as transformation on the space of states or density matrices. This prompts the natural question whether S [subject to condition (9)] lifts to an operator on the Hilbert space V. In other words, one asks whether there exists an operator $\widehat{S} : \ V \to V,\ \psi \mapsto \widehat{S}\psi$ such that

$$S(\rho) = \widehat{S} \circ \rho \circ \widehat{S}^{-1}. \tag{10}$$

The answer to this question turns out to be YES and is known as *Wigner's Theorem* (see [3] for an old and [4] for a modern proof). One has to distinguish between two cases. For a given S, the operator $\widehat{S}$ is either *unitary*,

$$\big\langle \widehat{S}\psi_f, \widehat{S}\psi_i \big\rangle = \langle \psi_f, \psi_i \rangle, \tag{11}$$

or *anti-unitary*,

$$\big\langle \widehat{S}\psi_f, \widehat{S}\psi_i \big\rangle = \overline{\langle \psi_f, \psi_i \rangle}. \tag{12}$$

Here, $\langle \psi_f, \psi_i \rangle$ denotes the Hermitian scalar product of the quantum Hilbert space V.

Having introduced the basics of symmetry operations on states, let us now turn to the dual side of physical observables. On general grounds, an operation $\rho \mapsto S(\rho)$ on states induces an operation $A \mapsto S'(A)$ on observables. This comes about because the pairing between states and observables is required to be invariant: the expectation value of the symmetry-transformed observable $S'(A)$ in the symmetry-transformed state $S(\rho)$ must be equal to the expectation value of A in the state ρ. In the realization of both A and ρ as operators on V, this forces that

$$S'(A) = \widehat{S} \circ A \circ \widehat{S}^{-1}. \tag{13}$$

Remark. It follows that a symmetry operation is automatically an automorphism of the algebra of observables: $S'(AB) = S'(A)S'(B)$.

Next comes an important inference. Recall that the density matrix ρ of any state (pure or mixed) in quantum mechanics is a semi-positive self-adjoint operator of unit trace. In particular, ρ has the property $\rho \geq 0$, and this property has to be preserved by symmetry transformations. By the principle of invariant pairing between states and observables, it follows that a symmetry operation takes a semi-positive observable A to another such observable:

$$A \geq 0 \Rightarrow S'(A) = \widehat{S} A \widehat{S}^{-1} \geq 0. \tag{14}$$

Now, any Hamiltonian H in quantum mechanics must be bounded from below (or else the theory would not have a ground state) to be acceptable. As an operator bounded from below, H can be made positive by shifting the zero on the energy axis. Applied to the Hamiltonian, Eq. (14) says that $H \geq 0$ remains positive under any symmetry transformation. Thus an operation (such as “chiral symmetry” or “sublattice symmetry”, see Sect. 4) that reverses the sign of the Hamiltonian does not qualify as a symmetry.

The set of symmetry operations we have described so far is a large set containing many “silly” operations that are of little practical consequence and interest. We shall now sharpen the symmetry concept by adding a restriction: by a symmetry (in the true and final sense) we mean an operation that leaves the quantum dynamics invariant (i.e., maps solutions to solutions). Then, to decide the question of symmetry or no symmetry, we need a model of quantum dynamics. For a prototype we may look at the Schrödinger equation of single-particle quantum mechanics,

$$\mathrm{i}\hbar\, \partial_t \psi = H \psi\, . \tag{15}$$

We see that if ψ is a solution of Eq. (15) and $\psi \mapsto \widehat{S}\psi$ is a unitary operation, then $\widehat{S}\psi$ is a solution if the operator $\widehat{S}$ commutes with the Hamiltonian:

$$\widehat{S} \circ H \circ \widehat{S}^{-1} = H. \tag{16}$$

In the case of an anti-unitary operation, $\widehat{S}\, \mathrm{i} \widehat{S}^{-1} = -\mathrm{i}$, we still have a symmetry if Eq. (16) holds, provided that one qualification is made: solutions with time running forward get transformed by $\widehat{S}$ to solutions with time running backward, as $\mathrm{i} \to -\mathrm{i}$ is to be compensated by $\partial_t \to -\partial_t$.

In summary, a symmetry in quantum mechanics is subject to two conditions: (i) acting on the space of states it preserves all transition probabilities, and (ii) acting on the space of observables it commutes with the Hamiltonian, or the generator of the quantum dynamics.

3 Quantum billiard with magnetic fluxes

While our focus in the main part of this lecture will be on anti-unitary symmetries of particle-hole type, let us begin with a few words on a famous (and non-controversial) cousin, namely time reversal T. By non-relativistic reduction of the time-reversal operation on Dirac spinors, T acts on spinful electrons as an antilinear operation with square $T^2 = -1$. The presence of the minus sign has an important consequence known as *Kramers degeneracy*, as follows. Let T be a symmetry ($TH = HT$). Then if ψ is an eigenvector of H, so is $T\psi$. Assuming the proportionality $T\psi \stackrel{?}{=} \lambda\psi$ ($\lambda \in \mathbb{C}$), one quickly gets a contradiction with $T\lambda = \bar{\lambda}T$ and $T^2 = -1$. It follows that ψ and $T\psi$ must be linearly independent. Thus they form a so-called Kramers pair $\{\psi, T\psi\}$ of degenerate energy eigenvectors. (As a brief aside, this kind of Kramers pair rose to prominence in recent activities on the quantum spin Hall effect.)
We now present an example showing that T-symmetry may have non-trivial consequences even in the absence of spin (when $T^2 = +1$). Consider the quantum chaotic billiard of a spinless free particle moving in a compact domain with von Neumann boundary conditions at the concave boundary. Let the billiard have fourfold rotational symmetry. Assume that the quantum particle is charged and that four magnetic flux insertions with alternating circulation reduce the rotational symmetry to a twofold one as shown in Fig. 1.
The group of unitary symmetries of this system is $\mathbb{Z}_2 = \{\mathrm{Id}, R_\pi\}$, consisting of the neutral element and rotation through π about the central point. In this simple example, the Hilbert space of a single particle decomposes into two subspaces: the R_π-even and the R_π-odd states.
The magnetic billiard also has two anti-unitary symmetries; these are time reversal followed by rotation through either $+\pi/2$ or $-\pi/2$ (to restore the magnetic circulation). Denote the first of these by $T' \equiv R_{\pi/2} \circ T$. Because time reversal commutes with space rotations, so does T'. Moreover, the anti-unitary operator T' squares to R_π and thus to $+1$ on the R_π-even and -1 on the R_π-odd sector. As a result, there exists a qualitative difference between the energy spectra for the two sectors: the R_π-even eigenstates generically come with non-degenerate energy eigenvalues, whereas the R_π-odd eigenstates organize into Kramers doublets. (Speaking the language of the Tenfold Way [5], the R_π-even sector is of symmetry type AI, while the R_π-odd sector is of symmetry type AII.)

4 “Symmetries” that aren’t symmetries

Having given a reasonably precise definition and a simple example of what we mean by a symmetry, we now address the real-life complication that “symmetry” is often used in a sloppy and even misleading sense.
The literature often speaks of gauge “symmetries”. This prompts us to emphasize that electromagnetic gauge invariance is <u>not</u> a symmetry in our sense. Rather, gauge invariance is simply a consistency condition, which arises because one prefers (for convenience, not by necessity) to work with gauge-dependent quantities, even though the physics is gauge-invariant and a formulation avoiding gauge fields altogether would in principle be feasible. (Indeed, mathematically

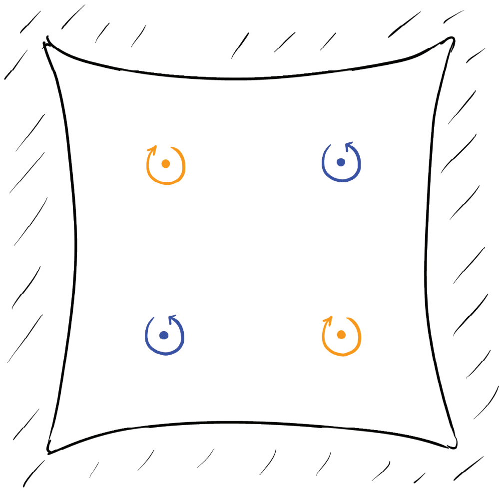

Fig. 1: *Quantum billiard with four magnetic flux insertions that alternate in circulation. The symmetry group is* $G = \mathbb{Z}_4 = \{T'^0,\ T'^1,\ T'^2,\ T'^3\}$ *generated by time reversal in combination with rotation by an angle of* $\pi/2$.

speaking, the wave functions of quantum mechanics are sections of a complex vector bundle and as such invariant under gauge transformations.) The crux of the matter can be explained by way of an informative analogy: in principle, a vector v in a vector space is invariantly defined, but in practice one expresses it by its components with respect to one basis or another:

$$v = \sum e_i\, v^i = \sum \tilde{e}_i\, \tilde{v}^i. \tag{17}$$

Choosing a fixed basis $\{e_i\}$ is like "fixing the gauge", and changing to another basis $\{\tilde{e}_i\}$ amounts to making a gauge transformation. Gauge invariance in this context is the simple fact that any expression with physical meaning can depend on the components v^i or $\tilde{v}^i$ only in a way that allows the expression to be rewritten in terms of the invariantly defined vector v (without invoking any basis). Once it is understood that gauge invariance is no more than a consistency condition, it is clear that:

- Gauge "symmetries" can never be broken, neither explicitly nor spontaneously (whereas symmetries can be broken).
- Gauge "symmetries" do not entail any conservation laws (whereas continuous symmetries do, by Noether's principle).

In particular, the conservation of electric charge should not be attributed to local $\mathrm{U}(1)$ electromagnetic gauge invariance (as is often done). We will not go deeper into this subject here, but refer our audience to the literature [6, 7].
Our next non-symmetry is chiral "symmetry", also known as a sublattice "symmetry" in the condensed matter context. So, assuming the setting of a lattice, consider a bipartite system whose Hilbert space V decomposes as $V = V_A \oplus V_B$. Let the Hamiltonian be off-diagonal with respect to that decomposition:

$$H = H_{A\leftarrow B} + H_{B\leftarrow A}\,. \tag{18}$$

If one defines a so-called sublattice transformation S by reversing the sign of the wave function, say, on the B-sublattice $S: \ \psi_A + \psi_B \mapsto \psi_A - \psi_B$, then S anticommutes with H:

$$SH = -HS. \tag{19}$$

This is an example of a chiral or sublattice "symmetry". Since S does not commute with the Hamiltonian, it is not a true symmetry in our sense. (It can, however, be turned into a symmetry in a suitable many-fermion setting; cf. Sect. 6.)
Another non-symmetry to be mentioned here is the particle-hole "symmetry" of the Bogoliubov-deGennes equations for a superconductor treated in the Hartree-Fock-Bogoliubov mean-field approximation. To spell that out, we invoke the formalism of second quantization (which we assume to be understood). Written in terms of the creation ($c^\dagger$) and annihilation (c) operators for some choice of single-particle basis, the second-quantized Hamiltonian is of the general form

$$H = \sum_{ij} \left(A_{ij} c_i^\dagger c_j + \tfrac{1}{2} B_{ij} c_i^\dagger c_j^\dagger + \tfrac{1}{2} C_{ij} c_i c_j \right), \quad \overline{A_{ij}} = A_{ji},\ C_{ij} = \overline{B_{ji}} = -C_{ji}\,. \tag{20}$$

The information contained in H can be recast as a matrix with block structure by organizing the expression as

$$H = \tfrac{1}{2} \sum_{ij} \begin{pmatrix} c_i^\dagger & c_i \end{pmatrix} \begin{pmatrix} A_{ij} & B_{ij} \\ C_{ij} & -\overline{A_{ij}} \end{pmatrix} \begin{pmatrix} c_j \\ c_j^\dagger \end{pmatrix} + \text{const.} \tag{21}$$

The resulting matrix h,

$$h \equiv \begin{pmatrix} A & B \\ C & -\overline{A} \end{pmatrix}, \tag{22}$$

or written in tensor-product notation,

$$h = A \otimes \tfrac{1}{2}(1+\sigma_3) + B \otimes \sigma_+ + C \otimes \sigma_- - \overline{A} \otimes \tfrac{1}{2}(1-\sigma_3), \tag{23}$$

is Hermitian ($h^\dagger = h$), and by construction it <u>always</u> satisfies the relation

$$h = -\Sigma_1 \, \overline{h} \, \Sigma_1\,, \quad \Sigma_1 = \mathbf{1} \otimes \sigma_1\,. \tag{24}$$

The matrix h is known the Bogoliubov-deGennes (BdG) "Hamiltonian", and the relation (24) is sometimes called the "particle-hole symmetry" of the BdG Hamiltonian – with the bizarre corollary that *every superconductor is particle-hole symmetric* (?!). In actual fact, neither is h a Hamiltonian, as it is not bounded from below in general, nor is the relation (24) a symmetry in the proper sense. Indeed, that relation is no more than a reflection of ($h = h^\dagger$ and) the canonical anticommutation (CAR) relations for fermion Fock operators:

$$c_i^\dagger c_j^\dagger + c_j^\dagger c_i^\dagger = 0 = c_i c_j + c_j c_i\,, \quad c_i^\dagger c_j + c_j c_i^\dagger = \delta_{ij}\,. \tag{25}$$

Since these CAR relations constitute an algebraic foundation that cannot ever be violated, the relation (24) should really be attributed to the *structure* of the theoretical formalism, not to any symmetry of the Hamiltonian.

5 The conundrum of charge conjugation

Having gone through a list of popular "symmetries" that aren't symmetries, we now turn to a true symmetry which, however, is not a particle-hole symmetry: charge conjugation.

Charge conjugation is a symmetry operation of the relativistic quantum theory of the Dirac field. It is one of the protagonists of the CPT-theorem, which states that any relativistic QFT with "reasonable" properties must be invariant under the combined operations of charge conjugation, parity transformation, and time reversal. A main stay in the realm of particle physics, charge conjugation has also come up in condensed matter physics, although it is of little relevance there. Let us spend a few words about it.

For a newcomer to the field, the situation with charge conjugation C may seem somewhat confusing. One the one hand, charge conjugation can be seen as an operation that transforms solutions of the Dirac equation to other solutions. As such, it is anti-unitary [8]. Indeed, the Dirac equation for a relativistic Dirac particle with mass m and momentum p is

$$\mathrm{i}\hbar\,\partial_t\psi = h\psi\,, \quad h = \beta mc^2 + c\sum \alpha_l\, p_l\,, \quad p_l = \frac{1}{\mathrm{i}}\frac{\partial}{\partial x_l} \tag{26}$$

(in zero electromagnetic field). A popular choice for the Dirac matrices β and α_l is

$$\beta = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}, \quad \alpha_l = \begin{pmatrix} 0 & \sigma_l \\ \sigma_l & 0 \end{pmatrix} \quad (l = 1, 2, 3).$$

For that choice, one checks that solutions $t \mapsto \psi(\cdot, t)$ of the Dirac equation map to solution $t \mapsto C\psi(\cdot, t)$ (with the time t still running forward) by the transformation

$$C\psi = \beta\alpha_2\overline{\psi}, \tag{27}$$

which is antilinear. On the other hand, authoritative texts [9] state that charge conjugation is a linear operation, hence a unitary symmetry of the quantized Dirac field theory.

What's the resolution of this conundrum? In short, it is necessary to distinguish between two closely related but different notions. The anti-unitary operation C in (27) is no more (and no less) than a symmetry of the Dirac equation, which for present purposes is best viewed as a classical field equation. To arrive at a satisfactory quantum theory, some processing needs to be done; in particular, one has to come up with a good Fock vacuum (the "Dirac sea" of filled negative-energy states) converting the sign-indefinite h of the Dirac equation into an operator bounded from below. It turns out that the appropriate process of quantization does not take the anti-unitary symmetry C of the Dirac equation (26) to an operation acting on the Fock-Hilbert space of the quantum theory. To obtain something meaningful, C must be composed with a second antilinear operation (essentially, the Dirac ket-to-bra bijection, a.k.a. Fréchet-Riesz isomorphism), so that the finished product, $\widehat{C}$, is complex linear and unitary.

For a quick diagnostic of what's wrong with C, one observes that $Ch = -hC$. Thus in view of our mantra that *symmetries always commute with the Hamiltonian*, we have to concede that (i) either h is not the Hamiltonian, or (ii) C cannot be a symmetry. We'll leave it at that here and refer to the Appendix for more detail.

The take-away message here is that charge conjugation, properly understood as an operation on Fock-Hilbert space, is a unitary transformation. A brief characterization is

$$(\rho, j) \xrightarrow{C} (-\rho, -j), \quad \big(E, B\big) \xrightarrow{C} \big(-E, -B\big). \tag{28}$$

In most (if not all) condensed matter settings of interest, at least one of the components E, B of the electromagnetic field will be present. Therefore, charge conjugation symmetries are of little relevance to condensed matter physics.
To summarize, particle-hole symmetry should not be confused with charge conjugation symmetry. The latter is unitary whereas the former is anti-unitary. The electromagnetic field (E, B) is sent by the latter to $(-E, -B)$, but by the former to $(-E, B)$.

6 Su-Schrieffer-Heeger model

Building on the non-example of chiral "symmetry", we now develop our first example of a true symmetry of particle-hole type. We adopt an informal style (relegating the more formal aspects to the following section) and first convey the basic and general idea.
Let us recycle from Sect. 4 the setting of a bipartite system (with sublattices A and B) and Hilbert space decomposition $V = V_A \oplus V_B$. We then second-quantize the Hamiltonian of Eq. (18) as an operator acting on the fermionic Fock space generated by V:

$$H = \sum_{ij} \Big(c_{Ai}^\dagger H_{Ai,Bj}\, c_{Bj} + c_{Bj}^\dagger H_{Bj,Ai}\, c_{Ai} \Big)\,. \tag{29}$$

Consider now transforming the operator algebra by an automorphism K which is defined by

$$K(c_{Ai}) = c_{Ai}^\dagger\,, \quad K(c_{Bj}) = -c_{Bj}^\dagger\,, \tag{30}$$

in conjunction with the property of being antilinear and involutive:

$$K(H_{Ai,Bj}) = \overline{H_{Ai,Bj}}\,, \quad K^2 = \mathrm{Id}. \tag{31}$$

It is easy to see that this operation K leaves the Hamiltonian invariant:

$$\begin{aligned} K(H) &= \sum_{ij} \Big(K(c_{Ai}^\dagger) K(H_{Ai,Bj}) K(c_{Bj}) + K(c_{Bj}^\dagger) K(H_{Bj,Ai}) K(c_{Ai}) \Big) \\ &= \sum_{ij} \Big(c_{Ai} H_{Bj,Ai} \big(-c_{Bj}^\dagger\big) - c_{Bj} H_{Ai,Bj}\, c_{Ai}^\dagger \Big) \quad = \ H, \end{aligned} \tag{32}$$

where in the last step we used the CAR relations to restore normal ordering, putting annihilation operators in the right and creation operators in the left position. K satisfies all the structural requirements posited in the opening section (i.e., it is an automorphism of the operator algebra and preserves all transition probabilities). Hence the relation $K(H) = H$ tells us that K is a symmetry of the many-fermion system with Hamiltonian H. Since K is antilinear, the symmetry is of anti-unitary type. We note that K is qualitatively different from the anti-unitary symmetry of time reversal T: the latter transforms creation operators amongst themselves and annihilation operators amongst themselves, whereas the former mixes/exchanges creation with annihilation operators. (For that reason, K was called a "mixing symmetry" in the foundational paper [5].)

Let us now take the general setting of a bipartite system and specialize it to a model actively studied in contemporary condensed matter physics. For this we associate one single-particle state with each site of a one-dimensional chain of sites labeled by the integers $n \in \mathbb{Z}$. The single-particle states on even sites ($n \in 2\mathbb{Z}$) span the subspace V_A, those on odd sites span V_B. The Hamiltonian is a kinetic energy of hopping between adjacent sites

$$H_{\text{kin}} = \sum_{n\in\mathbb{Z}} \left(t_{n+1,n}\, c_{n+1}^\dagger c_n + t_{n,n+1}\, c_n^\dagger c_{n+1} \right), \quad t_{n,n+1} = \overline{t_{n+1,n}}. \tag{33}$$

This Hamiltonian has the particle-hole symmetry $K(H_{\text{kin}}) = H_{\text{kin}}$ given by

$$K(c_n) = (-1)^n c_n^\dagger\,, \quad K(c_n^\dagger) = (-1)^n c_n\,, \quad K(\mathrm{i}) = -\mathrm{i}\,. \tag{34}$$

Notice that there is no condition on the hopping amplitudes $t_{n,n+1}$: these can be complex and even n-dependent in some random way and K will still be a symmetry.

We now take a closer look at translation-invariant systems with real hopping: $t_{n,n+1} \equiv -t_0 \in \mathbb{R}_+$. In that case, it is useful to transform to the momentum representation by introducing

$$a_k = \sum_{n\in\mathbb{Z}} \mathrm{e}^{-\mathrm{i}kn} c_n\,, \quad a_k^\dagger = \sum_{n\in\mathbb{Z}} \mathrm{e}^{+\mathrm{i}kn} c_n^\dagger \quad (k \in \mathbb{R}/2\pi\mathbb{Z}). \tag{35}$$

Note that these momentum-space operators satisfy the anticommutation relations

$$a_k^\dagger a_{k'} + a_{k'} a_k^\dagger = 2\pi\, \delta(k-k'). \tag{36}$$

The Hamiltonian $H \equiv H_{\text{kin}}$ of the translation-invariant system then takes the diagonal form

$$H = \int \frac{dk}{2\pi}\, \varepsilon(k)\, a_k^\dagger a_k\,, \quad \varepsilon(k) = -t_0 \cos k. \tag{37}$$

In the momentum basis, the operation K of particle-hole symmetry acts as

$$K(a_k) = a_{k+\pi}^\dagger\,, \quad K(a_k^\dagger) = a_{k+\pi}\,, \quad K(\mathrm{i}) = -\mathrm{i}\,. \tag{38}$$

Note that $K(H) = H$ holds more generally for any k-odd dispersion relation $\varepsilon(k\pm\pi) = -\varepsilon(k)$. The situation is illustrated graphically in Fig. 2 for the cosine band $\varepsilon(k) = -t_0 \cos k$. The figure also indicates the fact that the lift $\widehat{K}$ of K to the many-fermion Fock space leaves the free-fermion ground state invariant only if the cosine band is at half filling.

Finally, in order to arrive at the Su-Schrieffer-Heeger (SSH, [10]) model, one staggers the hopping amplitude:

$$0 < t_0 = t_{2n-1,2n} < t_1 = t_{2n,2n+1}. \tag{39}$$

Thus the chain alternates between weak bonds (with hopping amplitude t_0) and strong bonds (amplitude $t_1 > t_0$). The alternation reduces the group of translation symmetries from $\mathbb{Z}$ to $2\mathbb{Z}$

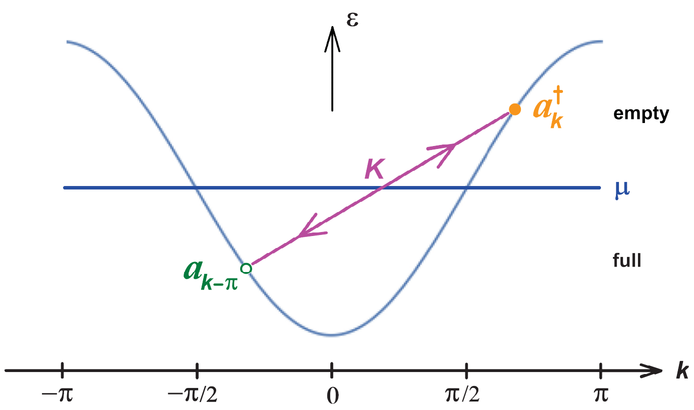

Fig. 2: *Particle-hole symmetry of the cosine band at half filling. The particle-hole transformation K exchanges the single-particle creation operator at momentum k with the single-hole creation operator at the shifted momentum $k \pm \pi$.*

(translation by two lattice sites). A good way to handle the situation is to double the unit cell in real space (and cut the Brillouin zone in momentum space in half). The single band of before then splits into two bands, and the staggered hopping opens an energy gap between the two. At half filling, the upper one is a *conduction* band, the lower one a *valence* band.

Let us now discuss some physical consequences of the particle-hole symmetry of the SSH model. When the Fermi energy lies in the gap between the conduction and valence bands, the SSH model is an example of a topological (band) insulator in one space dimension. Such systems have been under active investigation over the last two decades. More specifically, the SSH model represents a so-called *symmetry-protected topological phase* (SPT phase), with the protecting symmetry being the particle-hole symmetry K. Let us give a brief exposition.

By a principle known as *bulk-boundary correspondence*, the hallmark of any SPT phase is the appearance of gapless edge modes in a system with boundary (when there is something "topological" and non-trivial about the ground state of the bulk system without boundary). In the case of the SSH model, it is quite easy to exhibit the relevant effect. For that purpose, consider the SSH model on the half-space lattice $\mathbb{N} \cup \{0\}$ (i.e. the positive integers including zero). Let the bond between the boundary site $n = 0$ and its neighbor $n = 1$ be weak (t_0), the bond between $n = 1$ and $n = 2$ strong (t_1), and so on. Then it is easy to check that the operator

$$\psi = \sum_{n \geq 0} (-t_0/t_1)^n \, c_{2n}^\dagger \tag{40}$$

commutes with the SSH Hamiltonian H, and so does its adjoint $\psi^\dagger$:

$$[H, \psi] = 0 = [H, \psi^\dagger]. \tag{41}$$

Thus, when acting on the ground state of H (or any eigenstate, for that matter), the operators $\psi \, / \, \psi^\dagger$ create/annihilate an excitation of zero energy. Since $t_0 < t_1$, the magnitude $|-t_0/t_1|^n = \mathrm{e}^{-n/\xi}$ ($\xi > 0$) of the coefficients in the sum (40) decreases exponentially with increasing distance from the boundary $n = 0$. In other words, the operator ψ creates a zero mode exponentially localized near the system boundary. That's the gapless edge mode of the SSH model.

One may now ask: is the gapless edge mode a fluke/outlier unique to the SSH Hamiltonian H, or is there some robustness to the phenomenon, i.e., does the gapless feature survive when we deform H to, say, complex or even disordered hopping amplitudes? A detailed look into this question is outside the scope of the present 90 minute lecture. So, we just state the known outcome [11]: the gapless edge mode does remain in place as long as the deformation preserves particle-hole symmetry and leaves the energy gap of the bulk system open. Owing to their topological protection by symmetry, we can make the gapless edge disappear only by tuning the system through a phase transition where the energy gap of the bulk system closes.

Let us add a little precision to this story. One can imagine stacking a number N of identical SSH chains, all with boundary at $n = 0$. Then instead of one gapless edge mode we have N such modes. When we couple the chains within the parameter space of the free-fermion Hamiltonian, all N modes are known to remain gapless (as long as particle-hole symmetry is preserved and the bulk energy gap is kept open). What happens when electron-electron interactions are turned on (still preserving the particle-hole symmetry)? The answer is that interactions can change the number N of gapless edge modes without passing through a phase transition, but the change ΔN is always an integer multiple of 4. In other words, the topological classification of one-dimensional topological insulators with particle-hole symmetry (known as symmetry class AIII in the Tenfold Way) is $\mathbb{Z}/4\mathbb{Z}$. So, we can take away the message that the anti-unitary symmetry of particle-hole transformation has some utility!

7 Kitaev-Majorana chain

Another example of interest can be generated by varying the previous one. Let us take the Hamiltonian H_{kin} of Eq. (29) and add pair creation and annihilation terms:

$$H_{BDI} = H_{\text{kin}} + \sum_{n \in \mathbb{Z}} \left(\Delta_{n+1,n}\, c_{n+1}^\dagger c_n^\dagger + \Delta_{n,n+1}\, c_n c_{n+1} \right), \quad \Delta_{n,n+1} = \overline{\Delta_{n+1,n}}\,. \tag{42}$$

This extended Hamiltonian still has the particle-hole symmetry K of Eq. (34) for any choice of pairing amplitudes $\Delta_{n+1,n}$ (randomly depending on n). It can serve as the Hamiltonian for a superconductor (of spinless or spin-polarized electrons with zero chemical potential) in the Hartree-Fock-Bogoliubov (HFB) mean-field approximation.

Let us insert a quick comment on notation and language. The previous example (SSH) featured two symmetries: the $\mathrm{U}(1)$ symmetry behind particle-number conservation and a particle-hole symmetry K. In the present case, the $\mathrm{U}(1)$ symmetry is gone; this change modifies the symmetry class from AIII to BDI in the Tenfold Way [5].

The Hamiltonian H_{BDI} is very easy to analyze (and still informative) when the hopping and pairing amplitudes are chosen in a special translation-invariant way, so that

$$H_{BDI} \equiv H = t \sum_{n \in \mathbb{Z}} \left(c_{n+1} - c_{n+1}^\dagger \right) \left(c_n + c_n^\dagger \right). \tag{43}$$

We refer to this H as the Hamiltonian of the *Kitaev-Majorana chain*. We can characterize its ground state fully by the complex lines A_k of its quasi-particle annihilation operators. By

Fourier transformation these are

$$A_k = \mathbb{C}\alpha_k\,, \quad \alpha_k = \sum\nolimits_n \mathrm{e}^{\mathrm{i}kn}\left(\mathrm{i}\sin(k/2)\,c_n + \cos(k/2)\,c_n^\dagger\right).$$

Indeed, one easily checks that the α_k obey the commutation relation $[H,\,\alpha_k] = -t\alpha_k$ and thus lower the energy by $t > 0$. The state of lowest energy, known as the superconducting ground state in the HFB mean-field approximation, is the one annihilated by all these energy-lowering operators. Note that although α_k is double-valued as a function of $k \in \mathbb{R}/2\pi\mathbb{Z}$, the fiber bundle of complex lines $A_k = \mathbb{C}\alpha_k$ is well-defined. (Mathematically speaking, we are facing a line bundle in a non-trivial isomorphism or K-theory class. The "square-root nature" of the α_k signals the existence of a topological invariant which is robust against deformation and, by bulk-boundary correspondence, protects gapless edge modes of Majorana type.)
Application of the particle-hole transformation (34) to the annihilation operators α_k gives

$$K A_k K^{-1} = A_{\pi-k}\,. \tag{44}$$

This means that the superconducting ground state is particle-hole symmetric, as K transforms the set of quasi-particle annihilation operators amongst themselves. (It should be mentioned that this property gets lost when the chemical potential moves away from zero.)

8 Lifting ph-symmetry to Fock space

To achieve a satisfactory understanding of the workings of particle-hole symmetry K, we need its lift $\widehat{K}$ to the fermionic Fock space $\mathcal{F} = \bigwedge(V)$, i.e. a mapping $\widehat{K}\,:\ \mathcal{F} \to \mathcal{F}$ such that

$$K(A) = \widehat{K} \circ A \circ \widehat{K}^{-1}. \tag{45}$$

The lift $\widehat{K}$ can be described at various levels of sophistication. Here, to keep things simple, we shall restrict our setting to that of a finite-dimensional single-particle Hilbert space V. (This setting encompasses the SSH chain of finite length and the lowest Landau level of a quantum Hall system with finite area.) For the general case of infinite dimension, we refer our audience to the technical chapters of the review article [2].
The transformation K of a particle-hole symmetry is a concatenation of two operations: (i) a system-dependent transformation, say Γ, which arises already in the initial setup of a single-particle Hilbert space V, and (ii) a universal operation that we refer to as *particle-hole conjugation*. In the example of the SSH chain, Γ is the momentum shift $k \to k \pm \pi$ (or, more generally, the sublattice transformation $\psi_A + \psi_B \mapsto \psi_A - \psi_B$); while particle-hole conjugation is always the same, namely the operation, say $A \mapsto A^\flat$, of switching between creation and annihilation operators ($a_k \leftrightarrow a_k^\dagger$). We hasten to add that particle-hole conjugation differs from Hermitian conjugation (even though the two coincide on Fock operators: $a_k^\flat \equiv a_k^\dagger$). Indeed, particle-hole conjugation is an algebra automorphism whereas Hermitian conjugation $A \mapsto A^\dagger$ is an algebra *anti*-automorphism, viz.

$$(AB)^\flat = A^\flat B^\flat \quad \text{vs.} \quad (AB)^\dagger = B^\dagger A^\dagger. \tag{46}$$

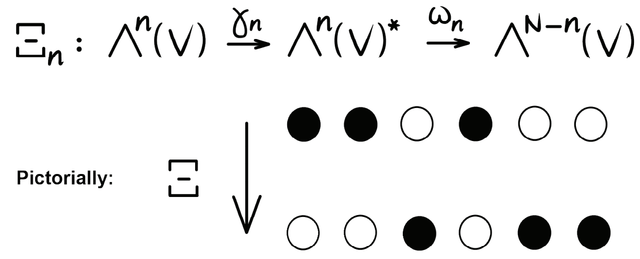

Fig. 3: *Transforming n particles into n holes, the antilinear operator Ξ_n of particle-hole conjugation is invariantly defined as the concatenation of the Fréchet-Riesz isomorphism γ_n with the wedge isomorphism ω_n. In the occupation-number representation with respect to any orthonormal single-particle basis, it takes filled orbitals to empty ones and vice versa. Consequently, particle-hole conjugation can never be a symmetry of any Fermi liquid.*

In brief, our story is going to unfold as follows. A short calculation [similar to that in Eq. (32)] shows that the universal operation of particle-hole conjugation sends every traceless Hermitian one-body Hamiltonian to its negative ($H^\flat = -H$). If the (non-universal) operator $\varGamma$ anti-commutes with the single-particle Hamiltonian (cf. chiral "symmetry"), then the concatenation K of $\varGamma$ and particle-hole conjugation leaves the Hamiltonian of the second-quantized theory invariant,

$$K(H) = \big(\varGamma(H)\big)^\flat = H, \tag{47}$$

and is thus a particle-hole symmetry of H. As we shall see, the mapping $A \mapsto A^\flat$ is necessarily complex antilinear. Typically, $\varGamma$ is complex linear (the case of charge conjugation being an exception). If so, the concatenation K is an anti-unitary operation. That will be the upshot of the current section.

After this overture with road map, we turn to the main agenda point here: constructing the Fock space lift, $\varXi : \mathcal{F} \to \mathcal{F}$, of particle-hole conjugation:

$$A^\flat = \varXi \circ A \circ \varXi^{-1}. \tag{48}$$

To give a very simple (if preliminary) description of $\varXi$, fix some orthonormal single-particle basis and work in the occupation number representation of $\mathcal{F}$, i.e., express states in Fock space as linear combinations of the basic states that have each single-particle state either occupied or empty. In this representation, the lift $\varXi$ of particle-hole conjugation flips all occupations ($0 \leftrightarrow 1$), see the lower part of Fig. 3. This is the rough picture of $\varXi$. From that picture, it is more or less clear that such $\varXi$ will swap creation with annihilation operators as desired. For the fine picture, we must take into account possible sign factors (due to the fermion algebra) that may have to be built into $\varXi$.

In an effort to be pedagogical, we defer the precise (and invariant) description of $\varXi$ until later and stick for now with the occupation number representation w.r.t. a fixed orthonormal basis, which we label by $i \in \{1, \ldots, N\}$. Adopting Dirac notation, we denote the Fock vacuum by $|\text{vac}\rangle$, and we express n-particle states with definite occupation numbers as

$$\varPhi_n = c_{i_1}^\dagger c_{i_2}^\dagger \cdots c_{i_n}^\dagger |\text{vac}\rangle. \tag{49}$$

Now since our Hilbert space V is assumed to be finite-dimensional, we can also speak about the state of total filling (i.e., the state with maximal particle number $N = \dim V,$ where every single-particle state is occupied). That state, $|\mathrm{sea}\rangle$, is only unique up to normalization and phase. So we make some fixed choice of $|\mathrm{sea}\rangle$. Then the particle-hole conjugate $\Xi\Phi_n$ is defined to be the state vector which is empty for each single-particle state occupied in the Φ_n of Eq. (49):

$$\Xi\Phi_n = c_{i_1} c_{i_2} \cdots c_{i_n} |\mathrm{sea}\rangle. \tag{50}$$

(Notice that the order of the index sequence remains unchanged.) This expression for $\Xi\Phi_n$ supersedes the hand-waving definition of Ξ ($\bullet \leftrightarrow \circ$) of before, by specifying the sign factors involved. It is good enough for most practical purposes. Nonetheless, as careful thinkers we wish to convince ourselves that Ξ actually exists as an invariantly defined (hence basis-independent) operation. That will occupy us for the rest of this section.

The invariant definition of Ξ uses two isomorphisms, which we refer to as "Fréchet-Riesz" and "wedge". We begin with the former. As a quantum mechanical Hilbert space, V is equipped with a Hermitian scalar product and hence an antilinear mapping

$$\gamma : \ V \to V^*, \quad v \mapsto \langle v, \cdot \rangle \,. \tag{51}$$

In physics, this mapping is also known as the Dirac ket-to-bra bijection, $|v\rangle \mapsto \langle v|$. It immediately generalizes to an antilinear Fock-space isomorphism

$$\gamma_n : \ \mathcal{F}_n \equiv \bigwedge^n(V) \to \mathcal{F}_n^* = \bigwedge^n(V)^* \cong \bigwedge^n(V^*) \tag{52}$$

by distributing γ over exterior products:

$$v_1 \wedge \cdots \wedge v_n \mapsto \gamma v_1 \wedge \cdots \wedge \gamma v_n \,. \tag{53}$$

This defines the Fréchet-Riesz isomorphism γ_n .

The definition of the wedge isomorphism is a little more involved. Recall that we have fixed a choice of fully occupied state vector $\psi_{\mathrm{sea}} \in \bigwedge^N(V)$. By taking an n-particle wave function ψ_n and wedging it (by exterior multiplication, which includes antisymmetrization) with an $(N{-}n)$-particle wave function ψ_{N-n}, we get some complex multiple of ψ_{sea}. Thus we may define

$$B : \ \mathcal{F}_n \otimes \mathcal{F}_{N-n} \to \mathbb{C}, \quad \psi_n \otimes \psi_{N-n} \mapsto B(\psi_n, \psi_{N-n}) \tag{54}$$

by the identification

$$\psi_n \wedge \psi_{N-n} \equiv B(\psi_n, \psi_{N-n})\, \psi_{\mathrm{sea}} \,. \tag{55}$$

The pairing B is complex bilinear and non-degenerate. Therefore, by a basic principle of linear algebra, it gives us a linear isomorphism

$$\omega_n : \ \mathcal{F}_n^* \to \mathcal{F}_{N-n}, \tag{56}$$

called *wedge isomorphism*, by the definition

$$\omega_n^{-1}(\psi_{N-n}) = B(\cdot, \psi_{N-n}) \in \mathcal{F}_n^* \,. \tag{57}$$

The finished product [12] of invariantly defined particle-hole conjugation ξ (restricted to the n-particle subspace $\mathcal{F}_n \subset \mathcal{F}$) then is the two-step process

$$\Xi_n : \ \mathcal{F}_n \xrightarrow{\gamma_n} \mathcal{F}_n^* \xrightarrow{\omega_n} \mathcal{F}_{N-n} . \tag{58}$$

Remark. Our informed audience may recognize some similarity between Ξ and the Hodge star operator $\star$ on differential forms in Riemannian geometry. The difference is that, given our setting in quantum mechanics, we are operating over $\mathbb{C}$ (not $\mathbb{R}$) and Ξ is antilinear (not linear).

9 Half-filled lowest Landau level

We have stressed that the universal operation of plain particle-hole conjugation can never be a symmetry of a Fermi liquid state or anything close to a free-fermion ground state. Recall the simple reason: particle-hole conjugation redistributes the population from the Fermi sea to the Fermi non-sea. Therefore, in order to stabilize the free-fermion or Fermi liquid ground state, a system-specific second transformation (Γ) must be applied.

Of course this no-go argument is void for interacting systems far from the free-fermion limit. The remainder of this lecture will be devoted to one rather spectacular example of a strongly correlated system – the half-filled lowest Landau level – where plain particle-hole conjugation does take the role of a symmetry.

To begin with history, it was Girvin [13] who pointed out the relevance of particle-hole conjugation for the physics of the quantum Hall effect (QHE). Recall that QHE takes place in a two-dimensional electron gas subject to a magnetic field. In the limit of a very strong magnetic field, one may project the single-electron Hilbert space to the lowest Landau level (LLL), say V_0 . Adopting the symmetric gauge for a homogeneous magnetic field of strength $B = -|B|\, dx \wedge dy$, the LLL wave functions are given by complex-analytic functions ψ of the dimensionless variable $z = (x+\mathrm{i}y)/\ell$. Here $\ell = \sqrt{\hbar/|eB|}$ is the magnetic length, and x, y are Cartesian coordinates for the plane $\mathbb{R}^2$. The Hermitian scalar product on V_0 is

$$\langle \psi, \psi' \rangle_{V_0} = \int d\mu(z)\, \overline{\psi(z)}\, \psi'(z), \quad d\mu(z) = \frac{|dx\, dy|}{2\pi\ell^2}\, \mathrm{e}^{-|z|^2/2}. \tag{59}$$

The description of V_0 is particularly simple in the case of a disk-shaped system. There, if the magnetic flux totals N flux quanta, V_0 is spanned by the N functions z^j for $j = 0, 1, \ldots, N-1$. The many-electron wave function $\psi_{\mathrm{sea}} \in \bigwedge^N (V_0)$ for the state of total filling can be expressed as a normalized Vandermonde determinant,

$$\psi_{\mathrm{sea}}(z_1, z_2, \ldots, z_N) = \mathcal{N}^{-1/2} \prod_{1 \le i < j \le N} (z_i - z_j). \tag{60}$$

Particle-hole conjugation $\Xi_n : \ \bigwedge^n (V_0) \to \bigwedge^{N-n} (V_0)$ here takes the concrete form

$$(\Xi_n \Psi)(z_{n+1}, \ldots, z_N) = \int \prod_{j=1}^{n} d\mu(z_j)\, \overline{\Psi(z_1, \ldots, z_n)}\, \psi_{\mathrm{sea}}(z_1, \ldots, z_n, z_{n+1}, \ldots, z_N). \tag{61}$$

Thus a complex-analytic and skew-symmetric function of n variables is particle-hole conjugated to another such function of $N-n$ variables. It is not difficult to see that Eq. (61) amounts to the same as (58).

9.1 Symmetry under particle-hole conjugation

The hallmark of quantum dynamics projected to the lowest Landau level (or any Landau level, for that matter) is that the kinetic energy of the charge carriers is totally quenched (if disorder or inhomogeneities in the background potential can be neglected), leaving no one-body component in the Hamiltonian of the bulk system. Now since the particle-hole conjugation operator Ξ sign-inverts the local charge density with respect to half filling, $\Xi Q(x) \Xi^{-1} = -Q(x)$, any residual two-body charge-charge (or current-current) interaction, in particular the Coulomb interaction, commutes with Ξ.

In the sequel, we assume our Hamiltonian H to be exactly particle-hole conjugation symmetric ($\Xi H = H \Xi$). Under that assumption, we would expect the ground state Ψ_0 to be particle-hole conjugation symmetric ($\Xi \Psi_0 \in \mathbb{C}\Psi_0$) at half filling. If so, we face an immediate complication from the free-fermion perspective: since Ξ exchanges filled single-particle levels with empty ones, a ground state invariant under Ξ cannot be of Fermi-liquid type (at least not in the original electron degrees of freedom).

The theoretical treatment of the subject took off in 1993 with the work of Halperin–Lee–Read (HLR, [14]), who did propose a Fermi-liquid ground state for the lowest Landau level at half filling. Converting electrons into composite fermions by a procedure called magnetic flux attachment, they argued that the latter could form a Fermi sea; the rough picture was that, by attaching two (fictitious) flux quanta to each electron, one cancels the background magnetic field on average, thus allowing the composite fermions to move as free fermions, at half filling. The technical step of flux attachment is carried out by introducing a fictitious gauge field, a, and adding to the field-theory Lagrangian a Chern-Simons term $a \wedge da$.

Although the HLR proposal was quite successful in fitting the observed phenomena, one bothersome issue remained: there exists no manifest particle-hole symmetry in the HLR field-theory Lagrangian. That's a serious worry because, as explained above, the Coulomb interaction projected to the lowest Landau level does have the particle-hole conjugation symmetry Ξ. Now much light and renewed interest has been thrown on the issue by a recent proposal of Son [15], which we summarize briefly.

9.2 Son's proposal and its physical meaning

Son [15] starts by observing that, for the purpose of developing a low-energy effective theory, one may realize the lowest Landau level as the subspace of zero modes of a massless Dirac fermion ψ, say with charge $q = +|e|$, in a homogeneous magnetic field:

$$S = \mathrm{i}\hbar c \int dt \int d^2 r \; \bar{\psi}\, \gamma^\mu \big(\partial_\mu - \mathrm{i} q A_\mu / \hbar\big) \psi + \cdots , \tag{62}$$

where the ellipses indicate residual interaction terms. In fact, adopting the symmetric gauge $A = |B|\big(x^1 dx^2 - x^2 dx^1\big)/2$ for $B = |B|\, dx^1 \wedge dx^2$, and choosing the gamma matrices $\gamma^0 = \sigma_3$, $\gamma^1 = \mathrm{i}\sigma_2$, $\gamma^2 = -\mathrm{i}\sigma_1$, one arrives at a Dirac Hamiltonian D of the form

$$D \propto \begin{pmatrix} 0 & \partial_z - \bar{z}/4 \\ \partial_{\bar{z}} + z/4 & 0 \end{pmatrix}, \quad z = \big(x^1 + \mathrm{i}x^2\big)\sqrt{|eB|/\hbar},$$

and the zero modes of this Hamiltonian,

$$\psi_0 = \begin{pmatrix} f(z) \\ 0 \end{pmatrix} \mathrm{e}^{-|z|^2/4}, \quad \partial_{\bar{z}} f(z) = 0,$$

are in bijection with the states spanning the lowest Landau level; see above.

For the relativistic system (62), one has command of the discrete symmetry operations of charge conjugation C, parity P, and time reversal T. The product CT is antilinear in second quantization and sends the electromagnetic field (E, B) to $(-E, B)$. Thus it is an anti-unitary symmetry of the massless Dirac fermion (62) in zero electric field E and for any magnetic field B. It is straightforward to check that CT coincides with our operation Ξ of particle-hole conjugation upon restriction to the zero-energy Landau level of the theory (62).

Let us emphasize once again that a Ξ-symmetric half-filled Fermi-liquid ground state does not exist, neither in the quantum Hall electron variables nor in the low-energy equivalent theory (62). In view of that no-go situation, one is motivated to look for a good change of variables by which to develop a Fermi-liquid description of some sort.

Assuming the starting point (62), Son [15] performs a so-called fermionic particle-vortex transformation to pass to a dual formulation (known as QED_3) by another Dirac field ξ coupled to a dynamical gauge field a (which coincides with the Chern-Simons dynamical gauge field a of HLR but for a pseudoscalar multiplicative factor, the Hall conductivity at half-filling):

$$S_{\mathrm{dual}} = 2\pi \mathrm{i}\hbar v_F \int dt \int d^2r\; \bar{\xi}\gamma^\mu \big(\partial_\mu - 2\mathrm{i}a_\mu/q\big)\, \xi + \int A \wedge da + \cdots , \tag{63}$$

where we adopt the convention $dx^0 = v_F\, dt$ and $\partial_0 = v_F^{-1}\partial_t$, as our physical system with characteristic speed v_F has only Galilean invariance (not Lorentz invariance). The dynamical gauge field $a = a_\mu dx^\mu$ is a gauge potential for the charge-current two-form $J = da$ of the two-dimensional electron gas. In particular, its time component a_0 is proportional to the orbital magnetization m of the 2D electron gas. The two-component spinor field ξ in (63) is called the Dirac composite fermion. It has the physical dimension of length^{-1}, and it is charge-neutral as it does not couple directly to the external gauge field A. In view of the fundamental duality between magnetic flux and electric charge, the coupling to the charge one-form a suggests that ξ carries an emergent magnetic flux. In fact, what ξ carries is *vorticity*, a quantity tied to the presence of magnetic flux; see below.

The half-filled lowest Landau level features a nonzero orbital magnetization $\langle m\rangle$, and according to (63) the magnetization $\langle m\rangle \sim \langle a_0\rangle$ as the time-component of a acts as a chemical potential

for the Dirac composite fermion ξ. Therefore one may well expect the latter to form a Fermi-liquid ground state by populating a Fermi sea up to the chemical potential $\langle m \rangle$.
Let us now take a closer look at the objects of the dual theory (63). For the benefit of our audience, we here adopt component notation (with respect to a Cartesian basis), assuming familiarity with epsilon tensors and the summation convention. We write

$$J = \tfrac{1}{2} J_{\mu\nu}\, dx^\mu \wedge dx^\nu, \quad J_{\mu\nu} = \epsilon_{\mu\nu\lambda} J^\lambda, \quad J^0 = \rho_{\text{exc}}\,, \quad J^l = j^l \ (l = 1, 2), \tag{64}$$

with j^l the components of the electric current vector field and $\rho_{\text{exc}} \equiv \rho$ the excess electric charge density with respect to half filling, and we put

$$a = a_\mu dx^\mu = m dt - \epsilon_{il}\, p^i dx^l. \tag{65}$$

The equation $J = da$ then splits into three equations:

$$\rho = -\partial_i p^i, \quad j^l = \epsilon^{il} \partial_i m + \partial_t p^l \quad (l = 1, 2). \tag{66}$$

The physical interpretation of the dynamical gauge field a should now be clear: p^i are the components an electric polarization vector field $\vec{p}$, and m is an orbital magnetization function for the 2D electron gas. These are determined only up to gauge transformations

$$m \mapsto m + \partial_t \phi\,, \quad p^l \mapsto p^l - \epsilon^{il} \partial_i \phi\,, \tag{67}$$

by a pseudoscalar function ϕ with the physical dimension of electric charge.
Turning to the Dirac composite-fermion field ξ, we expand on the statement that ξ carries vorticity by way of an emergent magnetic field. To start the argument, we observe that S_{dual} in (63) has a symmetry under global $\mathrm{U}(1)$ phase rotations (not to be confused with gauge transformations $a \mapsto a + d\phi$ and $\xi \mapsto \mathrm{e}^{2\mathrm{i}\phi/q}\xi$),

$$\xi(x) \mapsto \mathrm{e}^{\mathrm{i}\theta} \xi(x), \quad \bar{\xi}(x) \mapsto \mathrm{e}^{-\mathrm{i}\theta} \bar{\xi}(x), \tag{68}$$

which entails a conserved current:

$$\partial_\mu \Phi^\mu = 0, \quad \Phi^\mu = \bar{\xi} \gamma^\mu \xi. \tag{69}$$

Hence, the physical meaning of ξ hinges on the interpretation of the conservation law implied by $\partial_\mu \Phi^\mu = 0$. To uncover it, we introduce

$$b_{12} = \frac{2h}{q}\, \Phi^0\,, \quad e_i = \frac{2h}{q}\, v_F \epsilon_{ij} \Phi^j \quad (h = 2\pi\hbar,\ i = 1, 2). \tag{70}$$

The continuity equation (69) then takes the form

$$\partial_t b_{12} + \partial_1 e_2 - \partial_2 e_1 = 0\,, \tag{71}$$

which can be interpreted as Faraday's law of induction (written in components and transcribed to 2+1 space-time dimensions), with the consequence that the total space integral of b_{12} is independent of time. To reinforce the Faraday interpretation, we do an integration by parts, $\int A \wedge da = \int dA \wedge a$, and we decompose the electromagnetic field strength into its electric and magnetic parts: $dA = B + E \wedge dt$. We then see that the coupling $a_\mu \Phi^\mu$ between the dynamical gauge field a_μ and the conserved current Φ^μ enters into the dual action (63) as a shift:

$$B \to B + b\,, \quad E \to E + e \quad (b = b_{12}\, dx^1 \wedge dx^2,\ e = e_i\, dx^i). \tag{72}$$

Thus b and e are to be interpreted as *emergent magnetic and electric fields*. Moreover, taking the dual action (63) for granted, we see that the functional integral over a_0 pins the emergent magnetic field b to the external magnetic field B:

$$B_{\text{eff}} \equiv b + B = 0. \tag{73}$$

This constraint indicates that the conservation law $\int b \propto \int \Phi^0\, d^2r = \text{const}$ reflects the conservation of magnetic flux, $\int B = \text{const}$. Now, which conserved property of the electron gas is tied to the total external magnetic flux? There exists only one good answer to this question: the integrated *vorticity*, viz. the total number of zeroes in the one-body density matrix for the many-electron wave function; see [2] for more detail.

A follow-up comment concerns the factor of 2 in the expression $\nabla_\mu = \partial_\mu - 2\mathrm{i}\, a_\mu / q$ for the covariant derivative in (63). By canonical quantization of the Dirac composite field ξ, the presence of this factor means that single-particle excitations of ξ carry *two* emergent magnetic flux quanta $2h/q$ and hence two vortices due to the constraint (73). (The argument for that uses the Dirac quantization condition, stating that the lattice of quantized electric charges is h-reciprocal to the lattice of quantized magnetic fluxes.)

To add some perspective, it had long been appreciated that the LLL composite fermion is a composite object made from one electric charge quantum (electron) and two magnetic flux quanta (vortices). The traditional approach of HLR was to build the theory around the electron degrees of freedom and attach Chern-Simons magnetic flux as a subsidiary feature. The more recent approach of Son turned the emphasis around, by taking the vortex degrees of freedom (which carry magnetic flux but no electric charge) as primary and coupling them to fluctuations in the charge current J via the magnetization/polarization one-form a. Although the physical predictions turn out to be quite similar, the change of approach does make for some differences. For one, the number of composite fermions in the HLR approach is given by the number of electrons, whereas in Son's proposal it is given by half the number of magnetic flux quanta. For another, the realization of symmetries is different; see the next subsection.

9.3 Symmetry considerations

We now address the symmetry aspects of Son's proposal and especially the issue of particle-hole (conjugation) symmetry. Since particle-hole conjugation Ξ is realized in the (zero Landau level of the) Dirac fermion representation (62) by CT, we shall elucidate the corresponding operation CT in the dual representation (63).

As we are going to see, the operation of time reversal T is realized on the fermionic vortex field ξ in an unfamiliar way that makes the symmetry aspects quite striking. To spell them out, we begin by reviewing how T acts on the electromagnetic gauge field A and on the charge current J. The guiding principle here is that the field-matter interaction $A \wedge J$ must transform as a space-time density, so that

$$S_{\text{int}} = \int A \wedge J = \frac{1}{2} \int A_\mu dx^\mu \wedge J_{\nu\lambda}\, dx^\nu \wedge dx^\lambda = \int A_\mu J^\mu |d^3 x| \tag{74}$$

is invariant under <u>all</u> space-time diffeomorphisms including those that are orientation reversing, and is invariant under time reversal in particular. Now the two-form $J = da$ for the charge 3-current (in $2{+}1$ dimensions) is a *time-even* differential form, which means that it transforms under time reversal T by straight pullback: $J \mapsto +T^* J$. It then follows from $J = da$ and $T^* \circ d = d \circ T^*$ that the magnetization/polarization one-form $a = a_\mu dx^\mu$ is also time-even: $a \mapsto +T^* a$; in components we have

$$T: \; a_0(\mathbf{r}, t) \mapsto -a_0(\mathbf{r}, -t), \quad a_l(\mathbf{r}, t) \mapsto +a_l(\mathbf{r}, -t) \quad (l = 1, 2). \tag{75}$$

In contrast, the electromagnetic gauge field A is a *time-odd* one-form; thus it transforms under time reversal by negative pullback ($A \mapsto -T^* A$) or

$$T: \; A_0(\mathbf{r}, t) \mapsto +A_0(\mathbf{r}, -t), \quad A_l(\mathbf{r}, t) \mapsto -A_l(\mathbf{r}, -t) \quad (l = 1, 2). \tag{76}$$

(Of course, the opposite behavior of $J \mapsto +T^* J$ versus $A \mapsto -T^* A$ is just what is needed in order for $A \wedge J = A_\mu J^\mu |d^3 x|$ to transform as a scalar under time reversal.)
The sign-opposite transformation law for a as compared with A has a surprising effect. To see that most clearly, consider the first-quantized Hamiltonian h of the vortex field ξ in a given gauge field $a = m dt - p$:

$$h(p, m) = \hbar v_F \sum_{l=1}^{2} \sigma_l \left(\frac{1}{\mathrm{i}} \frac{\partial}{\partial x^l} + \frac{2}{q} p_l \right) - \frac{2\hbar}{q} m, \tag{77}$$

where $m = v_F a_0$ is the local magnetization, and $p = \sum p_l \, dx^l$ is the local polarization, treated here as a co-vector field (or form). To determine how T acts on ξ, one observes that if ξ is a solution of the Dirac equation $\mathrm{i}\hbar \, \partial_t \xi = h\xi$, then by time-reversal invariance (or equivariance) so is $T\xi$. There exist two different scenarios by which to realize that equivariance condition. The standard scenario is that T commutes with both h and $\mathrm{i}\hbar \, \partial_t$. Since ∂_t changes sign under $t \mapsto -t$, this means that T must be antilinear ($T\mathrm{i} = -\mathrm{i}\, T$). The second scenario for time-reversal symmetry of the Dirac equation is that T *anti*commutes with both $\mathrm{i}\hbar \, \partial_t$ and the Hamiltonian h. Given Eq. (77), it is clear that the time-even property of a, namely $T: \; m \mapsto -m$ and $p_l \mapsto +p_l$ ($l = 1, 2$), forces the latter scenario. Thus in the present instance, time reversal is realized as an operation that is complex *linear* ($T\mathrm{i} = +\mathrm{i}\, T$) and anticommutes with the vortex-field Hamiltonian h. Explicitly,

$$T: \; \xi(\mathbf{r}, t) \mapsto \sigma_3 \xi(\mathbf{r}, -t), \quad h(p, m) \mapsto \sigma_3 h(p, -m) \sigma_3 = -h(p, m). \tag{78}$$

This is how time reversal acts (in first quantization, i.e., viewing the Dirac equation as a classical field equation) on the fermionic vortex field ξ of the dual representation.
The other factor of the symmetry operation CT is charge conjugation C. We recall that the electromagnetic gauge field A transforms under C as $A \mapsto -A$. In order for the field-matter interaction $\int A \wedge da$ to be charge-conjugation invariant, the dynamical gauge field a must conform to the same transformation law as A (thus, $C: \; a \mapsto -a$). This is achieved by the antilinear transformation

$$C: \; \xi \mapsto \sigma_1 \bar{\xi}. \tag{79}$$

It follows that C anticommutes with the operator $\mathrm{i}\hbar\,\partial_t - h$ and hence reverses the sign of h:

$$C: \; h(p,m) \mapsto \sigma_1 \, \overline{h(-p,-m)} \, \sigma_1 = -h(p,m). \tag{80}$$

Next, we turn to the combined operation CT. By the properties of its factors C and T, the product CT is an antilinear symmetry of the first-quantized Hamiltonian:

$$CT: \; h(p,m) \mapsto \sigma_2 \, \overline{h(-p,m)} \, \sigma_2 = +h(p,m). \tag{81}$$

Note that the emphasis here is on *symmetry* (as opposed to antisymmetry $h \mapsto -h$). The field-matter interaction $\int A \wedge J$ for $J = da$,

$$\int A \wedge da = \int dA \wedge a = \int (B + E \wedge dt) \wedge (m\, dt - p), \tag{82}$$

augments the vortex-field Hamiltonian by a term

$$h_{\mathrm{int}} = \int (Bm - E \wedge p), \tag{83}$$

which is invariant under both T and C. The combined action of these operations on the electromagnetic field and the dynamical gauge field is

$$(E,B) \stackrel{CT}{\longrightarrow} (-E,B), \quad (p,m) \stackrel{CT}{\longrightarrow} (-p,m). \tag{84}$$

Now from our condensed-matter perspective, the electromagnetic field is to be regarded as a given background (not to be transformed). We then see that CT remains a symmetry for $E \equiv 0$ (zero external electric field) and any magnetic field B.
We can now deliver the (symmetry) punch line of Son's proposal. In the original formulation (62) by a massless Dirac fermion projected to the zero Landau level, the operator CT acted as particle-hole conjugation Ξ. Thus it exchanged particles and holes (or antiparticles) when acting on the Fock space of the second-quantized theory, thereby posing an obstruction to the existence of any Fermi-liquid ground state (with symmetry Ξ). Now in the dual representation (63) this obstruction has disappeared! Indeed, the operation CT on the fermionic vortex field ξ and the dynamical gauge field a is a proper symmetry of the Hamiltonian; when acting on the Fock space constructed by canonical quantization of the dual theory (63), it sends particles, or particle-like excitations of the vortex field, to particles, and it sends antiparticles to antiparticles. In Son's language, the Dirac composite fermion ξ is its own antiparticle. This, then, is how CT (alias particle-hole conjugation) may emerge as a symmetry of a Fermi-liquid ground state.

9.4 Microscopic picture of the composite fermion

To augment the field-theoretic symmetry considerations, which may seem quite abstract, we now offer a glimpse of the microscopic picture of the Dirac composite fermion. From experiment and theory one knows that the composite fermion at half filling ($\nu = 1/2$) is a charge-neutral excitation made, roughly speaking, from one electron and two fictitious magnetic flux quanta. How is this crude picture refined in view of Son's proposal?

For quantum Hall states and other systems with an energy gap where the quantum adiabatic theorem holds, one knows that the adiabatic insertion of a magnetic flux line (in 3D, or flux point in 2D), with circulation equal to that of the strong magnetic background field, expands the electron gas radially outward from the point of insertion. (We expect this to be still true for the gapless system of the half-filled lowest Landau level.) The adiabatic flux insertion gives rise to spectral flow resulting in a zero (or vortex) of the many-electron wave function. Adopting the symmetric gauge with respect to the insertion point, z_0 , one can express the effect of flux insertion as multiplication by the operator

$$U_{z_0} = \prod\nolimits_j (z_j - z_0), \tag{85}$$

where z_j ($j = 1, 2, ...$) are the electron coordinates of the holomorphic representation; see the very beginning of this Section. In the occupation-number representation $\mathbf{n} = \{n_0, n_1, n_2, ...\}$ w.r.t. the single-particle basis $(z - z_0)^l \mathrm{e}^{-|z-z_0|^2/4}$ ($l = 0, 1, 2, ...$), the effect of the vortex operator U_{z_0} is a shift $\mathbf{n} \mapsto \{0, n_0, n_1, \ldots\}$ leaving the zero orbital ($l = 0$) vacant. The vacancy amounts to a local charge deficit: for filling fraction ν, that deficit is $-\nu e$. Thus, the electric charge of a vortex excitation at half filling is $-e/2$.

Now from the particle-hole conjugation symmetry of the half-filled LLL, we expect the vortex-type excitation of charge $-e/2$ to be accompanied by its antiparticle, the "antivortex". Centered at position z_1, the antivortex is created by the particle-hole conjugate operator

$$U^\flat_{z_1} = \varXi U_{z_1} \varXi^{-1}. \tag{86}$$

By construction, an antivortex excitation carries the opposite charge ($+e/2$) and the same energy, provided that the ground state is particle-hole symmetric. Doing a short calculation in the occupation-number representation of above, one finds that $U^\flat_{z_1}$ has the same effect as U_{z_1} but for one characteristic difference: while U_{z_1} leaves the zero orbital (w.r.t. z_1) always empty, the particle-hole conjugate $U^\flat_{z_1}$ leaves it always occupied. Thus $U^\flat_{z_1}$ creates a vortex with charge deficit $-e/2$ centered at z_1 and at the same time adds an electron in the zero orbital at z_1, thereby producing a total charge increment of $+e/2$.

Given the basic building blocks of U_{z_0} and $U^\flat_{z_1}$, Wang and Senthil (WS, [16]) proposed to think of Son's composite fermion as a (more or less loosely) bound state of one vortex and one antivortex. Indeed, (i) there is an attractive electric force between a vortex and an antivortex, (ii) the total charge of a bound pair at $\nu = 1/2$ is zero, and (iii) the bound vortex-antivortex pair is its own antiparticle as the vortex component gets transformed under particle-hole conjugation to the antivortex component; cf. Fig. 4.

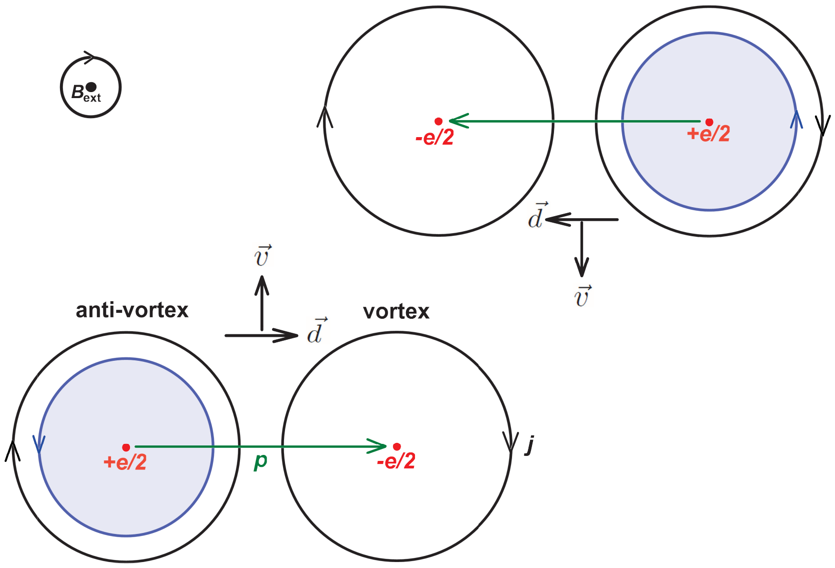

Fig. 4: *Semiclassical picture at $\nu = 1/2$ of a composite fermion (left-hand side) and its particle-hole conjugate (right-hand side). The composite fermion is an electric dipole made from a vortex with positive charge $-e/2$ and an antivortex with negative charge $+e/2$. In a state of motion, the velocity $\vec{v}$ and the dipole moment $\vec{d}$ form an orthogonal pair positively oriented with respect to B_{ext} .*

The composite fermions of observable consequence are those in a state of rapid motion (at the Fermi speed of the composite Fermi liquid). These carry an electric dipole moment $\vec{d}$ proportional to the relative position $z_1 - z_0$, and the dipole vector $\vec{d}$ is perpendicular to the velocity $\vec{v}$ of the motion – a phenomenon known as “spin-momentum” locking and anticipated in early work by N. Read [17]. Assuming the limit of large separation $|z_1 - z_0|$ between the vortex and antivortex constituents, WS argued that the process of adiabatically transporting a composite fermion around the Fermi surface, or rotating its k-vector through 2π, gives a Berry phase of π. (The latter had been recognized as an important ingredient [18, 15] for the correct phenomenology of the composite Fermi liquid.) It should be added that while the whole scenario sounds very plausible, its details still await confirmation by rigorous work.

Appendix

A Charge conjugation explained

In order to appreciate the symmetry of charge conjugation, one needs some understanding of the process of canonical quantization of the Dirac field. The latter requires three pieces of basic input: (i) the complex phase space W of classical solutions of the free Dirac equation, (ii) an invariantly defined symmetric bilinear form $B : \ W \otimes W \to \mathbb{C}$ (in order to construct the canonical anticommutation relations for the Dirac field), and (iii) a compatible complex structure J on the space $W_{\mathbb{R}}$ of real fields (to construct the proper ground state upon which to build the Fock space of physical electron and positron states of the Dirac field).

In this appendix, we focus on the first input, (i), as this already provides plenty of insight into the intricacies of charge conjugation. A key point is that one should think of the (complex) Dirac field as an object with two parts to it: a spinor part ψ that solves the free Dirac equation

$$D\psi = 0, \quad D = -\mathrm{i}\hbar\gamma^\mu \partial_\mu + mc, \tag{87}$$

and a *co-spinor* part $\widetilde{\psi}$ that solves the adjoint equation,

$$\widetilde{\psi} \circ D = 0. \tag{88}$$

(You may think of the spinor as a column vector and of the co-spinor as a row vector.) In other words, the complex phase space for the Dirac field is a direct sum $W = V \oplus \widetilde{V}$ of two subspaces: that of the spinor solutions ($\psi \in V$), and that of the co-spinor solutions ($\widetilde{\psi} \in \widetilde{V}$). The elements of the complex phase space W of solutions are pairs: $\big(\psi, \widetilde{\psi}\big) \in V \oplus \widetilde{V}$.

Now the Dirac operator D is formally adjoint to $D^\dagger = \gamma^0 D \gamma^0$. Therefore, the co-spinor $\widetilde{\psi} = \psi^\dagger \gamma^0$ solves the adjoint equation (88) if the spinor ψ solves Eq. (87). We say that the complex vector space W has a real subspace $W_{\mathbb{R}}$ spanned by the "real" solutions $\big(\psi, \widetilde{\psi}\big) = \big(\psi, \psi^\dagger \gamma^0\big)$.

To construct the quantum theory of the Dirac field, one has to work with the complex phase space W of solutions, as follows. One separates the spinor solutions ψ into their positive-frequency and negative-frequency parts, $\psi = \psi_+ + \psi_-$, by a so-called mode expansion (here indicated in schematic notation):

$$\psi_+ = \int d^3k\, u(k)\, \mathrm{e}^{-\mathrm{i}\omega_k t}, \quad \psi_- = \int d^3k\, v(k)\, \mathrm{e}^{+\mathrm{i}\omega_k t} \quad (\omega_k > 0). \tag{89}$$

This defines a decomposition $V = V_+ \oplus V_-$. The same procedure applied to the co-spinor solutions defines a decomposition $\widetilde{V} = \widetilde{V}_+ \oplus \widetilde{V}_-$. (We remark that by the bilinear form B not here specified, one has the duality $V_\pm^* \cong \widetilde{V}_\mp$.)

By the process of canonical quantization one turns elements of (more precisely: the coefficients appearing in the mode expansion of)

- V_+ into electron creation operators,
- V_- into positron annihilation operators,

- $\widetilde{V}_+ \cong V_-^*$ into positron creation operators,
- $\widetilde{V}_- \cong V_+^*$ into electron annihilation operators.

The Fock space of the quantized Dirac field then is the exterior algebra

$$\mathcal{F} = \bigwedge \big(V_+ \oplus \widetilde{V}_+\big) = \bigwedge \big(V_+ \oplus V_-^*\big). \tag{90}$$

Given this information, we can explain how charge conjugation works.
The operator $\psi \mapsto C\psi = \gamma^2 \bar{\psi}$ of Eq. (27) defines a mapping

$$C: \; V_\pm \to V_\mp \,, \tag{91}$$

which swaps positive-frequency and negative-frequency solutions. Thus, upon quantization, it takes an electron creation operator $\psi \in V_+$ and turns it into a positron annihilation operator $C\psi \in V_-$ (and vice versa). Such an operator does not act on the Fock space $\mathcal{F}$ of physical states. To obtain an operator that does act on $\mathcal{F}$, one needs to compose C with the mapping, say C', from spinor solutions to co-spinor solutions:

$$C': \; V_\mp \to \widetilde{V}_\pm \,, \quad \psi \mapsto \psi^\dagger \gamma^0. \tag{92}$$

Note that C' involves complex conjugation (via †) and hence switches between positive-frequency and negative-frequency solutions. Both C and C' are antilinear, so their concatenation $\widehat{C} = C' \circ C \, : \; V_\pm \to \widetilde{V}_\pm$ is complex linear. Altogether, $\widehat{C}$ maps electron creation operators to positron creation operators (and likewise for the annihilation operators). That's the unitary symmetry $\widehat{C}$ of charge conjugation of the relativistic Dirac field. (We re-iterate that $\widehat{C}$ is not a good model for anything of much relevance in condensed matter physics.)

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