NRG with Bosons

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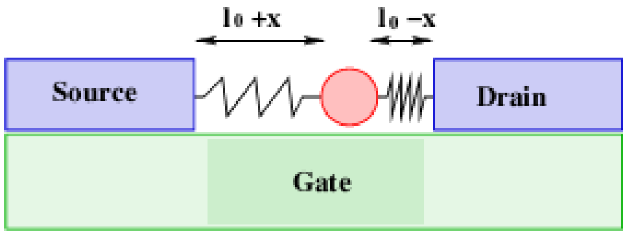
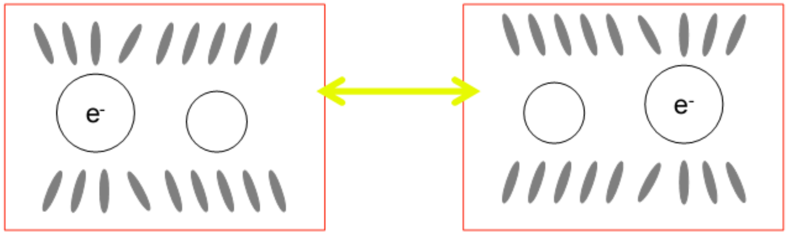
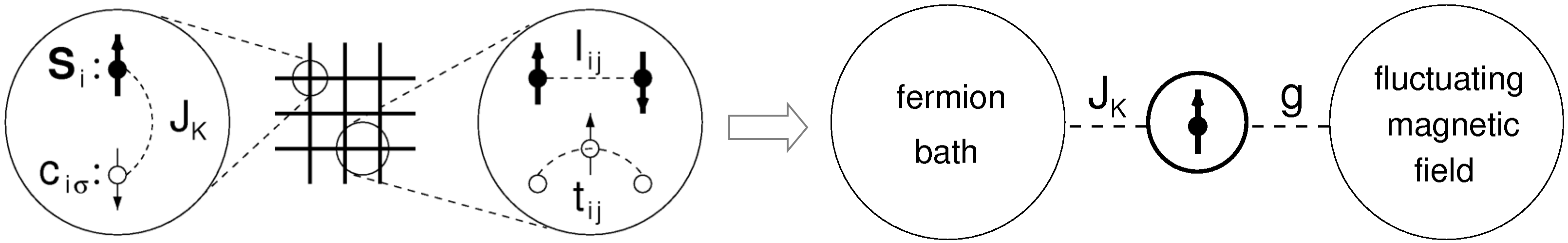


Al-Hassanieh et al., PRL (2005)



S. Tornow (unpub.)



Si et al., Nature (2001)

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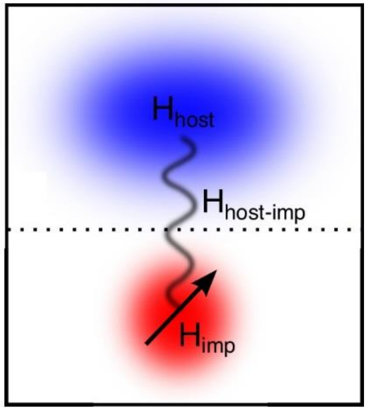
Outline

- Quantum impurity problems couple a local degree of freedom to an extended, noninteracting host:

$$H = H_{\text{host}} + H_{\text{host-imp}} + H_{\text{imp}}$$



- The NRG provides controlled nonperturbative solutions of problems involving fermions (T. Costi's talk).
- Extension of the NRG to problems involving bosons …
  - Inclusion of local bosons – Anderson-Holstein model
  - Formulation for bosonic hosts – spin-boson model
  - Combined approach for fermionic + bosonic hosts – Bose-Fermi Kondo model

I. Review: Fermionic NRG

- The NRG was developed for problems with fermionic hosts, e.g.,

$$H_{\text{Kondo}} = H_{\text{host}} + J\,\mathbf{S}_{\text{imp}} \cdot \mathbf{s}_{\text{host}}\left(\mathbf{r}_{\text{imp}}\right),$$

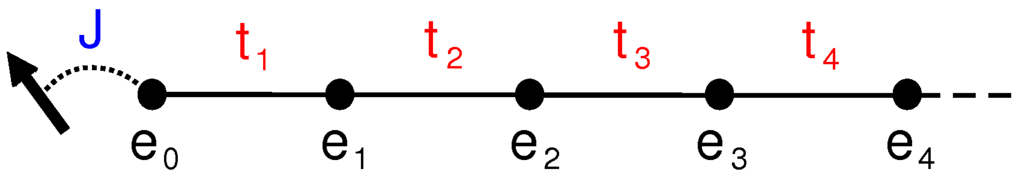
  where

$$H_{\text{host}} = \sum_{k,\sigma} \varepsilon_k\, c_{k\sigma}^{\dagger}\, c_{k\sigma}\,.$$

- A fundamental challenge of the Kondo model is the equal importance of spin-flip scattering of band electrons on every energy scale ε on the range $-D \le \varepsilon \le D$.
- Poor man's scaling attempts to tackle this, but it is perturbative in the renormalized Kondo coupling and thus limited to temperatures $T > T_K$ (A. Nevidomskyy's talk).
- The NRG was conceived to reliably reach down to $T = 0$.

Chain mapping of any host

- Any noninteracting host can be mapped exactly to a tight-binding form on one or more semi-infinite chains:



  - Start with $|f_0\rangle =$ host state entering $H_{\text{host-imp}}$
  - Since $i\hbar \frac{d}{dt}|\Psi\rangle = H|\Psi\rangle$,

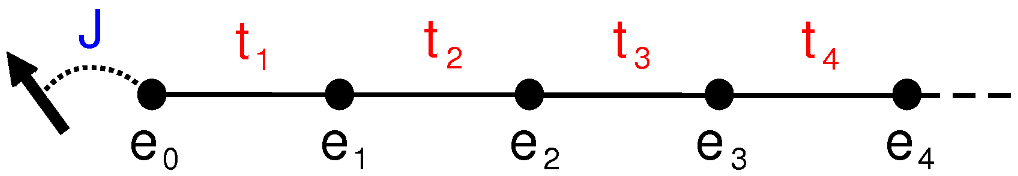
    reach only host states given by repeated action of H_{host}
  - Lanczos (1950): $H_{\text{host}}|f_0\rangle = e_0|f_0\rangle + t_1|f_1\rangle$

$$H_{\text{host}}|f_1\rangle = e_1|f_1\rangle + t_1|f_0\rangle + t_2|f_2\rangle$$

$$H_{\text{host}}|f_2\rangle = e_2|f_2\rangle + t_2|f_1\rangle + t_3|f_3\rangle \qquad \text{etc}$$

Chain mapping of a conduction band

- Any noninteracting host can be mapped **exactly** to a tight-binding form on one or more semi-infinite chains:



- The conduction band in the Kondo model maps to

$$H_{\text{host}} = \sum_{\sigma} \sum_{n=0}^{\infty} \left[e_n f_{n\sigma}^{\dagger} f_{n\sigma} + t_n \left(f_{n\sigma}^{\dagger} f_{n-1,\sigma} + \text{H.c.} \right) \right] + \text{decoupled part}$$

- Since the basis grows by a factor of 4 for each chain site, we would like to diagonalize H on finite chains. **But …**
  - Coefficients e_n, t_n are all of order the half-bandwidth.
  - No useful truncations: Ground state for chain length L is not built just from low-lying states for chain length $L-1$.

NRG's key feature: Band discretization

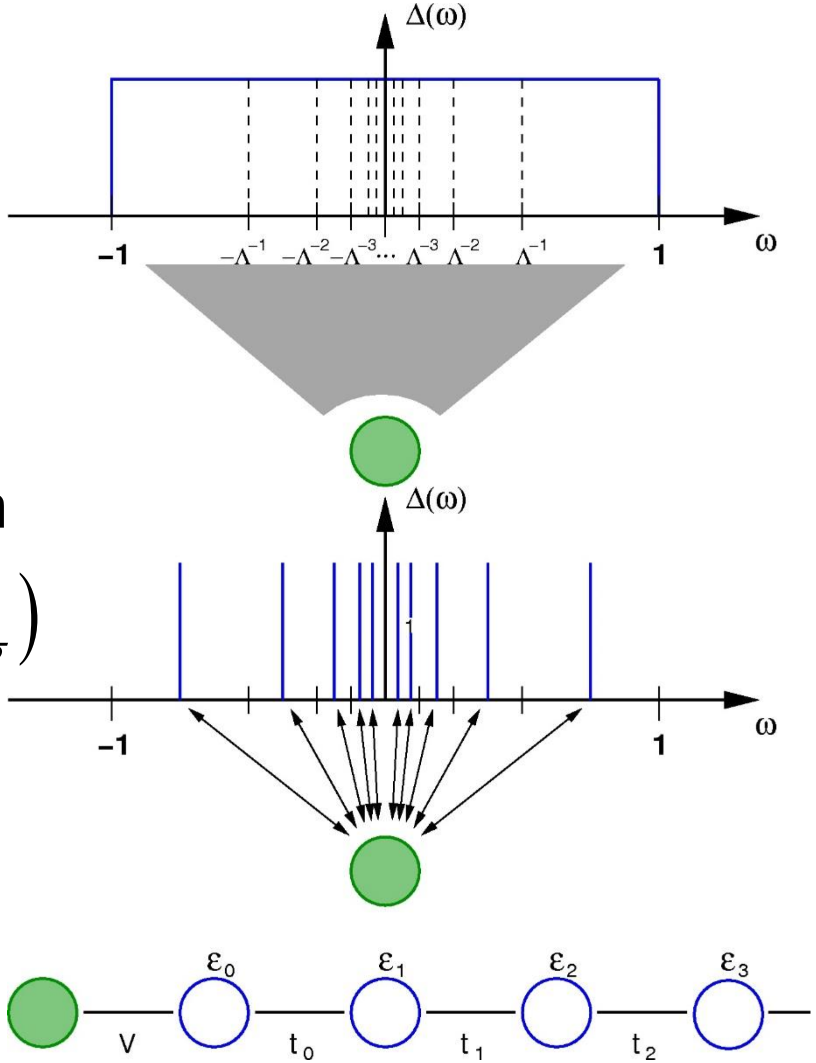
- Wilson (~1974) introduced a logarithmic discretization of the conduction band: $\Lambda > 1$

- Approximation: The impurity couples to just one state per bin

$$H_{\text{host},\Lambda} = \sum_{\sigma} \sum_{m=0}^{\infty} \omega_m \left(a^{\dagger}_{m\sigma} a_{m\sigma} - b^{\dagger}_{m\sigma} b_{m\sigma} \right)$$

$$\omega_m = \tfrac{1}{2}\left(1 + \Lambda^{-1}\right)\Lambda^{-m} D$$

- Now apply Lanczos to the discretized band:



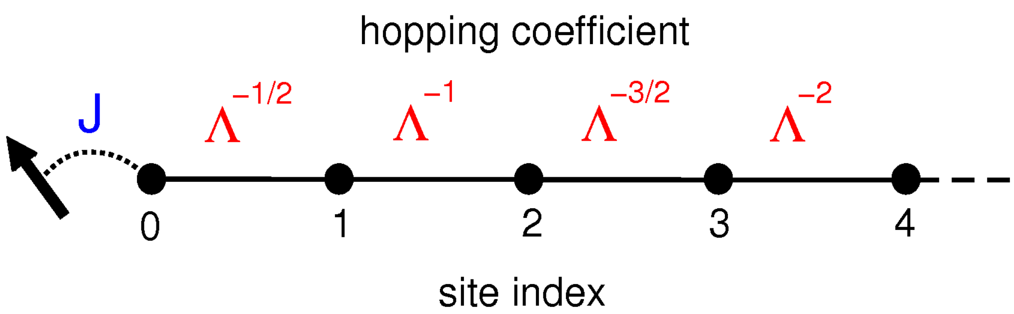
Bulla et al. (2008)

NRG iterative solution

- Wilson's artificial separation of bin energy scales $\propto \Lambda^{-m}$ gives exponential decaying tight-binding coefficients:

$$H_{\text{host},\Lambda} = \sum_{\sigma} \sum_{n=0}^{\infty} \left[e_n f_{n\sigma}^{\dagger} f_{n\sigma} + t_n \left(f_{n\sigma}^{\dagger} f_{n-1,\sigma} + \text{H.c.} \right) \right] \quad |e_n|, t_n \leq cD\Lambda^{-n/2}$$

(not a Λ^{-n} decay!)

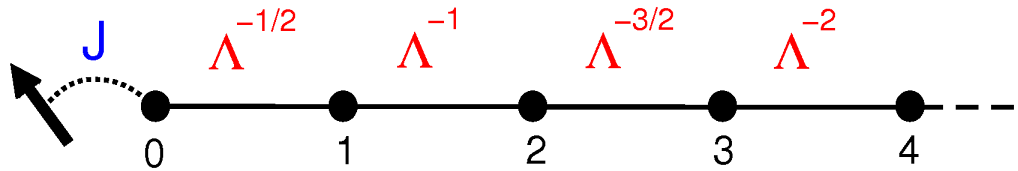


- Allows iterative solution on chains of length $L = 1, 2, 3, \ldots$
  - Ground state for chain length L is mainly built from low-lying states for chain length $L-1$.
  - Thus, can truncate the Fock space after each iteration.

NRG iterative solution

- Start with $H_0 = H_{\text{imp}} + H_{\text{host-imp}}\left(f_{0\sigma}^{\dagger}, f_{0\sigma'}\right) + e_0 f_{0\sigma}^{\dagger} f_{0\sigma}$.

  e.g., Kondo: $H_{\text{imp}} \to 0$; $H_{\text{host-imp}} \to 2J\, \mathbf{S}_{imp} \cdot \sum_{\sigma,\sigma'} f_{0\sigma}^{\dagger} \tfrac{1}{2} \tau_{\sigma\sigma'} f_{0\sigma}$; $e_0 \to 0$ (p-h symm.)



- Extend: $H_N = H_{N-1} + \sum_{\sigma} \left[e_N f_{N\sigma}^{\dagger} f_{N\sigma} + t_N \left(f_{N\sigma}^{\dagger} f_{N-1,\sigma} + \text{H.c.} \right) \right]$

  Diagonalize in a product basis $\left|E\right\rangle_{N-1} \otimes \left| b\left(f_{N\sigma}^{\dagger}, f_{N\sigma'}\right) \right\rangle$.

  Truncate to $N_s \approx 100 - 1000$ eigenstates of lowest energy.

- Repeat until reach a scale-invariant RG fixed point:

  spectrum of $H_N = \Lambda^{1/2} \times$(spectrum of H_{N-1}).

What does NRG give?

- The solutions of $H_{\text{host},\Lambda}$ can be used to calculate the value X_{host} of a bulk property in the pure host (without impurity).
- Solutions of

$$H = H_{\text{host},\Lambda} + H_{\text{host-imp}} + H_{\text{imp}}$$

give the value X_{total} in the full system with the impurity.

- Both X_{host} and X_{total} vary strongly with the discretization Λ.
- **<u>But</u>** the value of

$$X_{\text{imp}} = X_{\text{total}} - X_{\text{host}}$$

varies only weakly with Λ.

- Can use $2 \le \Lambda \le 10$ to estimate the physical $(\Lambda = 1)$ value.

II. Fermionic NRG with Local Bosons

- Now want to extend the method to problems with bosons.
- Simplest: Impurity couples to a single local boson mode (e.g., an optical phonon).
- Example: the Anderson-Holstein model

$$H_{\text{host}} = \sum_{\mathbf{k},\sigma} \varepsilon_k \, c^\dagger_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma} ,$$

$$H_{\text{host-imp}} = \frac{V}{\sqrt{N_k}} \sum_{\mathbf{k},\sigma} \left(d^\dagger_\sigma c_{\mathbf{k}\sigma} + \text{H.c.} \right),$$

$$H_{\text{imp}} = \varepsilon_d n_d + U n_{d\uparrow} n_{d\downarrow} + \omega_0 a^\dagger a + \lambda (n_d - 1)(a + a^\dagger)$$

Holstein coupling of impurity charge to oscillator displacement x

Anderson-Holstein model

$$H = H_{\text{Anderson}} + \omega_0 a^\dagger a + \lambda(n_d - 1)(a + a^\dagger)$$

1. Haldane (1977) proposed the model to describe mixed-valent rare-earth compounds, e.g. $CeAl_3$, $YbAl_2$.

   Local boson describes fast changes in $3d$ charge distribution in response to slower fluctuations in $4f$ occupancy, e.g. Ce $4f^0\ (4+) \leftrightarrow 4f^1\ (3+)$.

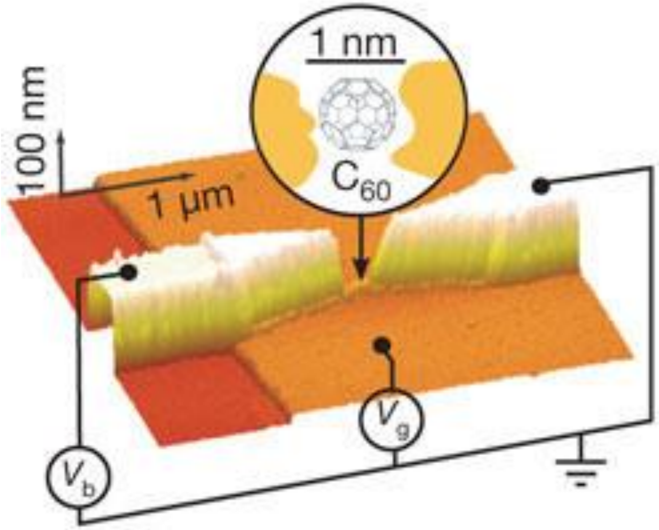
2. It is also a one-impurity version of a model (Anderson, 1975) for enhanced superconductivity due to "negative-U" centers in amorphous semiconductors.

   Local boson describes oscillations of a covalent bond length.

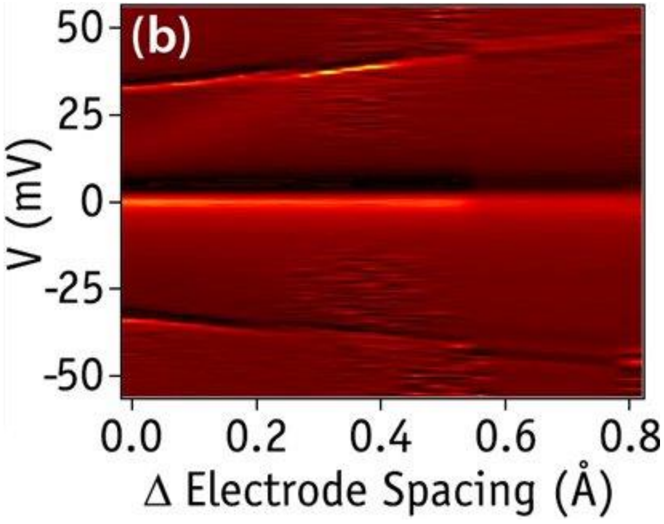
Anderson-Holstein model

$$H = H_{\text{Anderson}} + \omega_0 a^\dagger a + \lambda(n_d - 1)(a + a^\dagger)$$

3. Over the last 20 years, the model has been applied to quantum dots (Li et al., 1995, and many since) and single-molecule devices.



Roch et al., Nature (2008)



Parks et al., PRL (2007)

- "Anderson-Holstein" name: Hewson and Meyer (2002).

Decoupled limit

- Let's specialize to the symmetric case $\varepsilon_d = -U/2$.
  Then can rewrite

$$H_{\text{imp}} = \tfrac{1}{2}U\left(n_d - 1\right)^2 + \omega_0\, a^\dagger a + \lambda\left(n_d - 1\right)\left(a + a^\dagger\right).$$

- Consider the decoupled limit of zero hybridization $V = 0$.
  Now n_d is fixed and can use a displaced oscillator mode

$$b = a + \left(\lambda / \omega_0\right)\left(n_d - 1\right)$$

to write

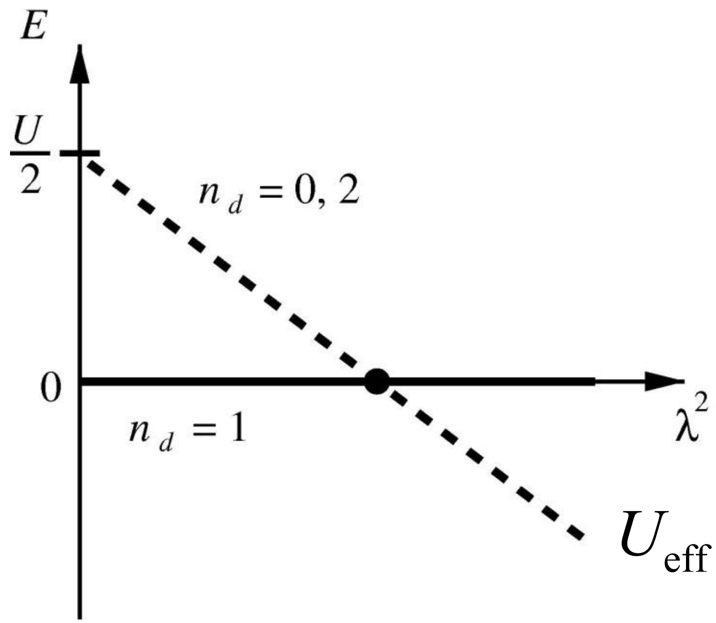
$$H_{\text{imp}} = \tfrac{1}{2}U_{\text{eff}}\left(n_d - 1\right)^2 + \omega_0\, b^\dagger b$$

where

$$U_{\text{eff}} = U - 2\lambda^2 / \omega_0.$$

Decoupled limit

- Since only states with $n_d \neq 1$ can benefit from the bosonic coupling, get a reduction in the effective on-site repulsion:



$$U_{\rm eff} = U - 2\lambda^2 / \omega_0.$$

- For $\lambda > \lambda_0 = \sqrt{\omega_0 U / 2}$, decoupled impurity has a charge-doublet ground state $n_d = 0, 2$.

Decoupled limit

- How many bosons are there in the ground state?
- From $$H_{\text{imp}} = \tfrac{1}{2} U_{\text{eff}} (n_d - 1)^2 + \omega_0 b^\dagger b$$

  it is obvious that in the displaced oscillator basis

$$\langle b^\dagger b \rangle_{GS} = 0.$$

- What about the original bosons $a = b - (\lambda / \omega_0)(n_d - 1)$?

$$\begin{aligned}\langle a^\dagger a \rangle_{GS} &= \langle b^\dagger b \rangle_{GS} - (\lambda/\omega_0)(n_d - 1)\langle b + b^\dagger \rangle_{GS} + (\lambda/\omega_0)^2 (n_d - 1)^2 \\ &= (\lambda/\omega_0)^2 (n_d - 1)^2.\end{aligned}$$

- $P(n_a)$ is a Poisson distribution with mean $(\lambda/\omega_0)^2$ and standard deviation λ/ω_0.

Full problem

- For $V > 0$, n_d is no longer fixed. Tunneling of an electron to/from the band creates & destroys a cloud of bosons as the oscillator adjusts to the new dot occupancy:

$$V \to V\, \exp\{(\lambda/\omega_0)(b - b^\dagger)\}$$ [Lang & Firsov (1962)].

- Adiabatic limit $\omega_0 \ll \Delta = \pi\rho(\varepsilon = 0)\, V^2$: Bosons can't adjust to changes in n_d, so don't affect the Kondo physics of Anderson model.
- Instantaneous limit $\omega_0 \gg \Delta$: Bosons are always relaxed w.r.t. instantaneous value of n_d. For $\omega_0 \gg U$, recover Anderson model physics with $U \to U_{\text{eff}} = U - 2\lambda^2/\omega_0$ and, for $U_{\text{eff}} < 0$, $\Delta \to \Delta_{\text{eff}} = \Delta\, e^{-(\lambda/\omega_0)^2}$.
- The most interesting regime for quantum dots, $\lambda \gg \Delta$ and $U \gg \omega_0$, is not susceptible to algebraic analysis.

NRG treatment of local bosons

Hewson & Meyer (2002)

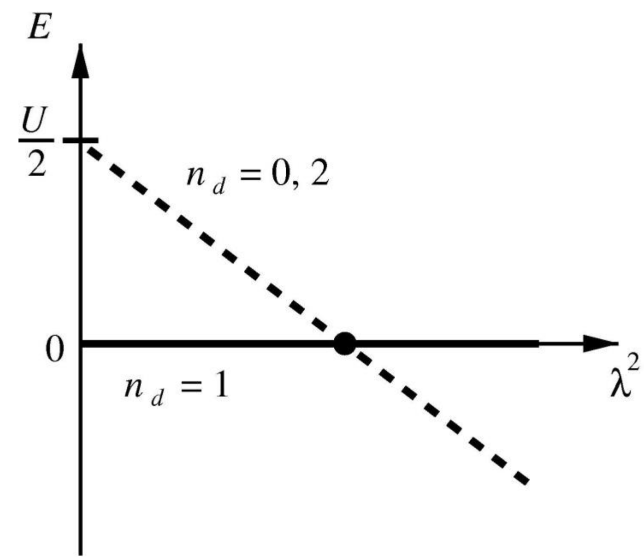
- Since a local boson has a finite energy ω_0, it can be included in
$$H_{\text{imp}} = \varepsilon_d n_d + U n_{d\uparrow} n_{d\downarrow} + \omega_0 a^\dagger a + \lambda(n_d - 1)(a + a^\dagger),$$
leaving untouched both $H_{\text{host-imp}}$ and H_{host}.

- Now H_{imp} has an infinite-dimensional Fock space that precludes exact diagonalization.

- But if we can find the right finite subset of eigenstates of
$$H_0 = H_{\text{imp}} + H_{\text{host-imp}}\left(f_{0\sigma}^\dagger, f_{0\sigma'}\right) + e_0 f_{0\sigma}^\dagger f_{0\sigma}$$
we should be able to use the conventional NRG approach to incorporate the conduction-band degrees of freedom.

NRG treatment of local bosons

- Based on the decoupled limit, expect the system to relax after each change in n_d toward a ground state with
$$\langle a^\dagger a \rangle_{GS} = (\lambda/\omega_0)^2 (n_d - 1)^2.$$
- To capture these states, Hewson & Meyer used a bosonic basis consisting of eigenstates of $a^\dagger a$ spanning
$$0 \le n_a \le n_{\max} = 4(\lambda/\omega_0)^2.$$
- To reach
$$U_{\text{eff}} = U - 2\lambda^2/\omega_0 \approx 0,$$
need
$$n_{\max} \approx 4U/\omega_0 \gg 1.$$



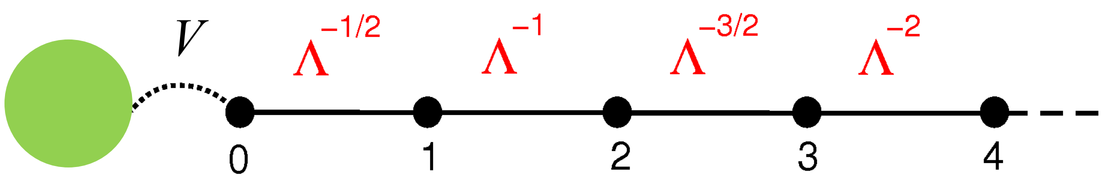
NRG treatment of local bosons

- In summary, can follow standard NRG iterative procedure

$$H_N = H_{N-1} + \sum_{\sigma} \left[e_N f_{N\sigma}^{\dagger} f_{N\sigma} + t_N \left(f_{N\sigma}^{\dagger} f_{N-1,\sigma} + \text{H.c.} \right) \right]$$

with a more complicated H_0 whose basis has dimension

$$4 \times 4 \times 4U / \omega_0 .$$



- This poses no fundamental problem for $U/\omega_0 \sim 10$ (say), which allows access to the most interesting regime:

$$\lambda \gg \Delta \text{ and } U \gg \omega_0 .$$

Results: Spin Kondo to charge Kondo crossover

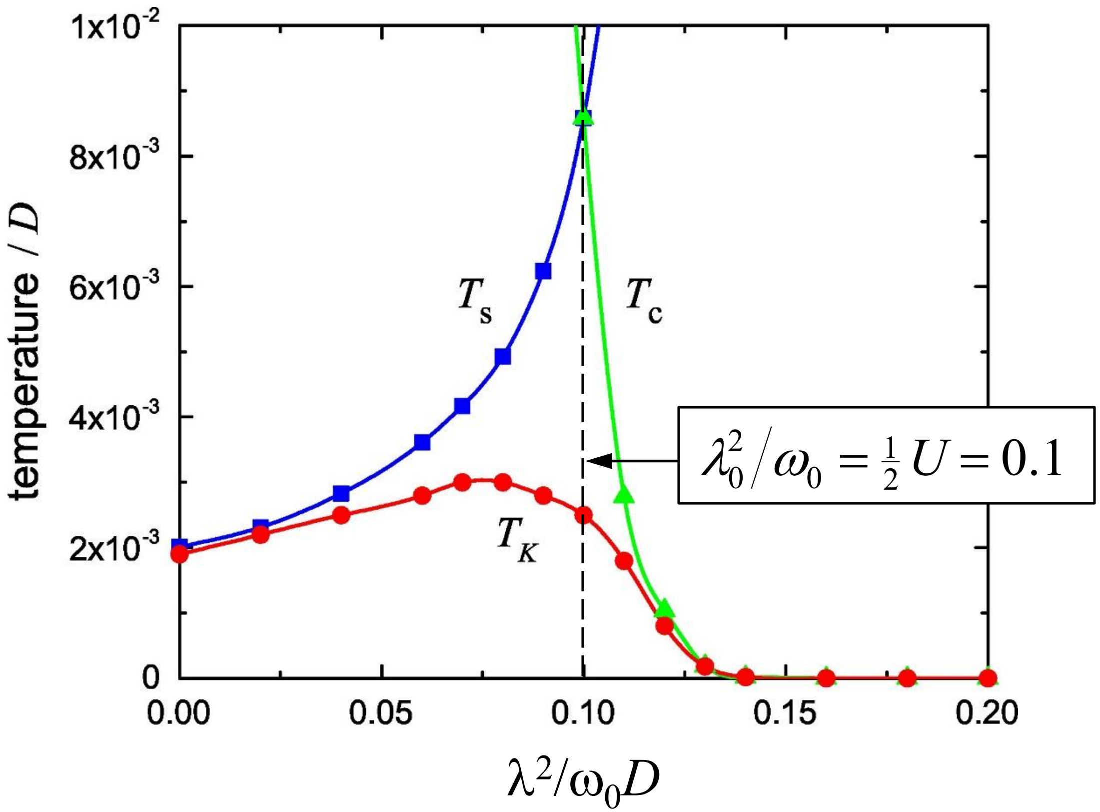
Conventional spin Kondo effect evolves smoothly with increasing λ into a charge Kondo effect where the impurity charge is collectively screened by band electrons.

Plot shows:

$$T_{\mathrm{s,c}} \sim 1/\chi_{\mathrm{s,c}}(T=0)$$

T_K from entropy

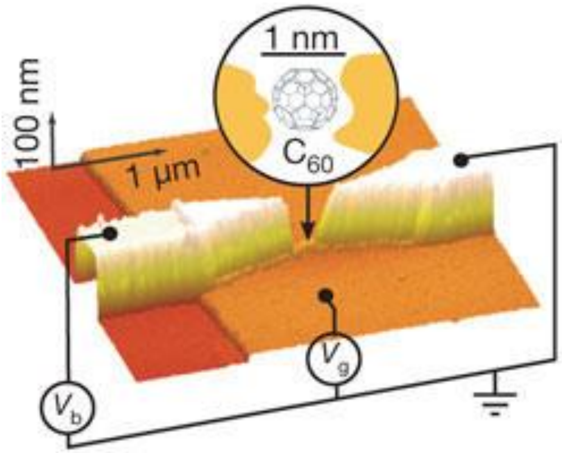
Note rapid fall of T_K for $\lambda > \lambda_0$

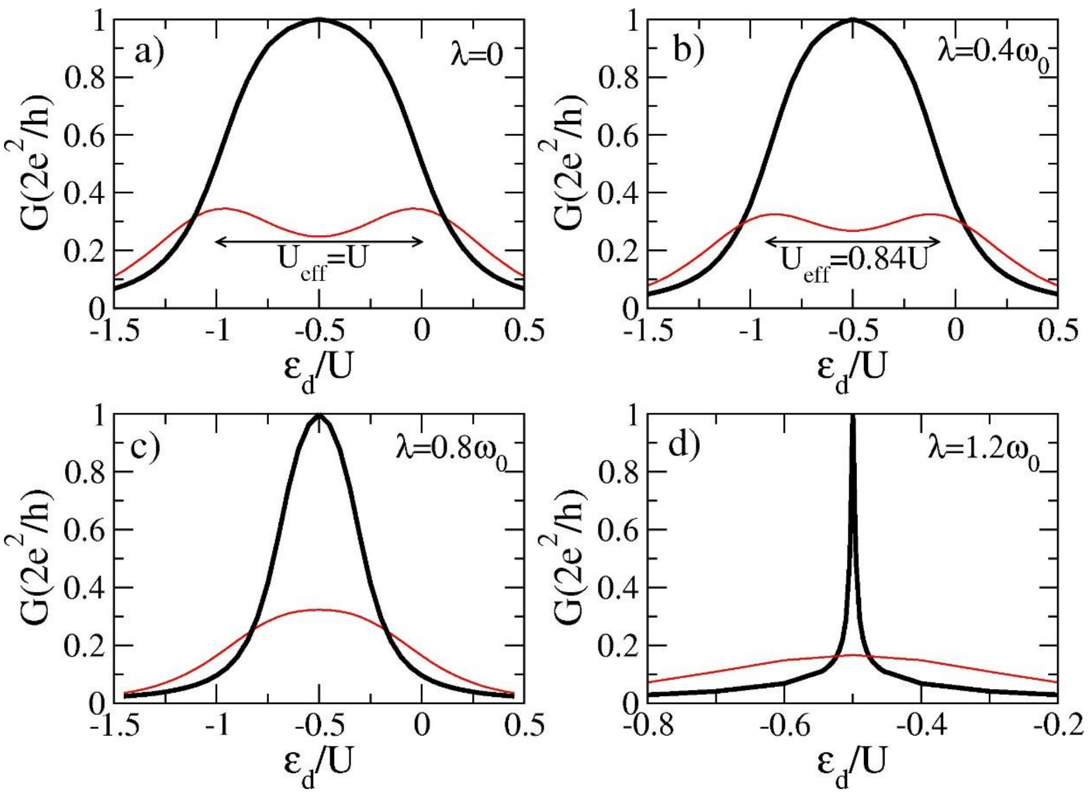


Results: Spin Kondo to charge Kondo crossover

In a quantum-dot, linear conductance

$$G(T=0)=\frac{2e^2}{h}\,\pi\Delta\rho_d(\omega=0)=\frac{2e^2}{h}\sin^2\frac{\pi\langle n_d\rangle}{2}$$





Gate voltage V_g

tunes model ε_d

Results: Spin Kondo to charge Kondo crossover

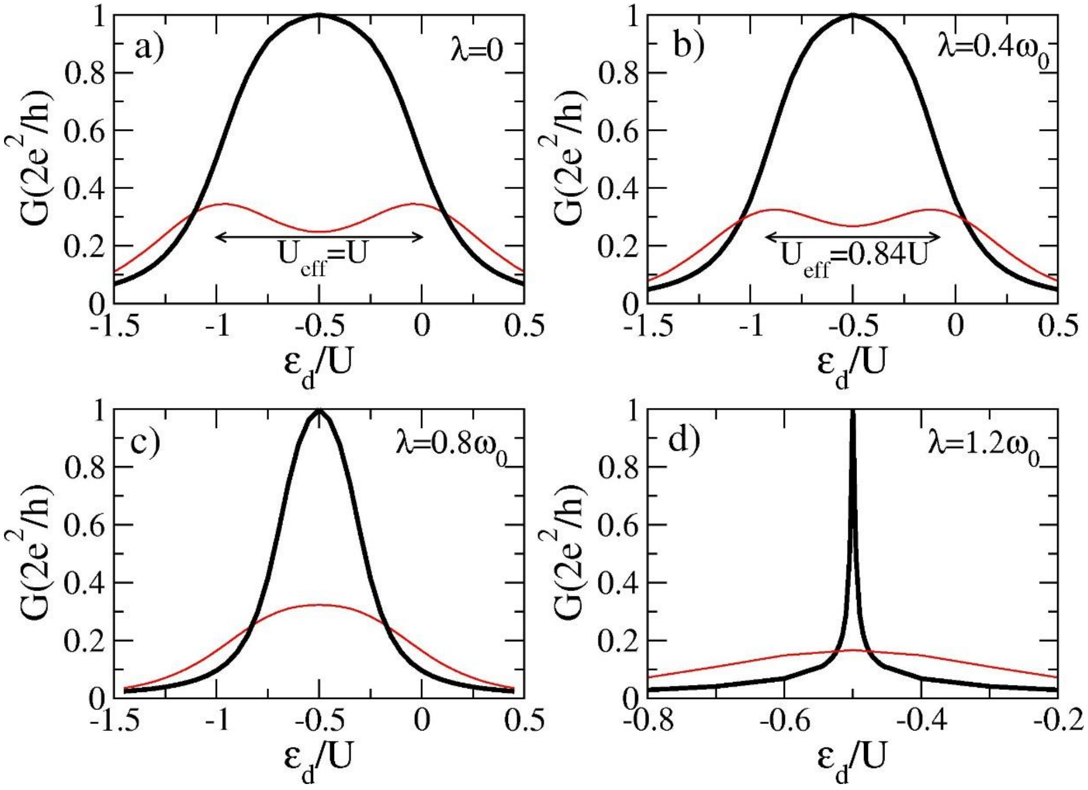
In a quantum-dot, linear conductance

$$G(T=0)=\frac{2e^2}{h}\pi\Delta\rho_d(\omega=0)=\frac{2e^2}{h}\sin^2\frac{\pi\langle n_d\rangle}{2}$$

For $\lambda > \lambda_0$ ($= \omega_0$ here) charge doublet g.s. is split by $|U+2\varepsilon_d|$.

Kills Kondo effect for

$$|U+2\varepsilon_d| > T_K(\varepsilon_d=-U/2).$$



II. Bosonic NRG

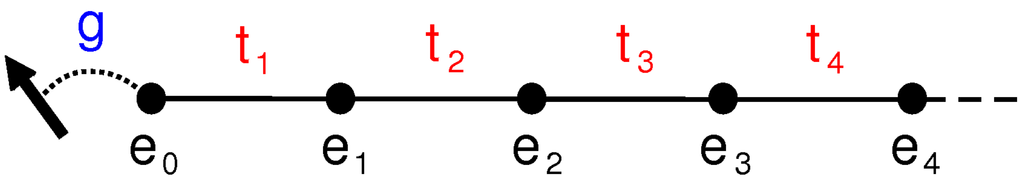
- Goals: Nonperturbative solutions of impurity problems with a gapless continuum of bosonic modes:

$$H_{\text{host}} = \sum_q \omega_q a_q^\dagger a_q \,.$$

- Typically, impurity couples to oscillator displacements:

$$H_{\text{host-imp}} = \hat{O}_{\text{imp}} \sum_q \lambda_q \left(a_q + a_q^\dagger \right).$$

- Again seek energy separation to allow iterative solution:



- Every bosonic chain site has an infinite basis, so …

  basis choice and state truncation likely to be crucial.

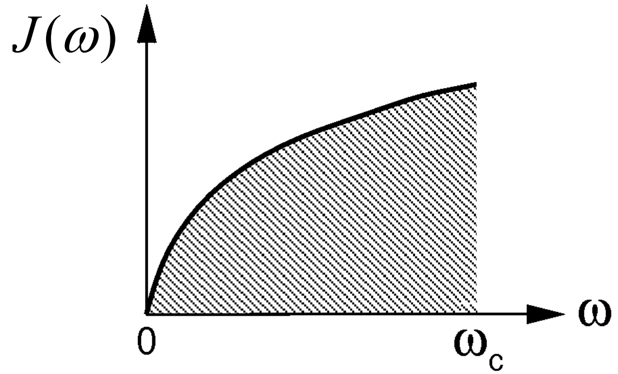
Spin-boson model

- A canonical model for coupling of a local degree of freedom to a dissipative environment:

$$H_{\text{host}} = \sum_q \omega_q a_q^\dagger a_q \,, \qquad H_{\text{imp}} = -\Delta S_x \,,$$

$$H_{\text{host-imp}} = \frac{1}{\sqrt{N_q}} S_z \sum_q \lambda_q \left(a_q + a_q^\dagger \right).$$

- ω_q and λ_q enter only through the bath spectral function



$$J(\omega) = \frac{\pi}{N_q} \sum_q \lambda_q^2 \, \delta\left(\omega - \omega_q\right)$$

$$= 2\pi \alpha \, \omega_c \left(\omega / \omega_c\right)^s \quad \text{for} \quad 0 < \omega < \omega_c \,.$$

NRG discretization and chain mapping

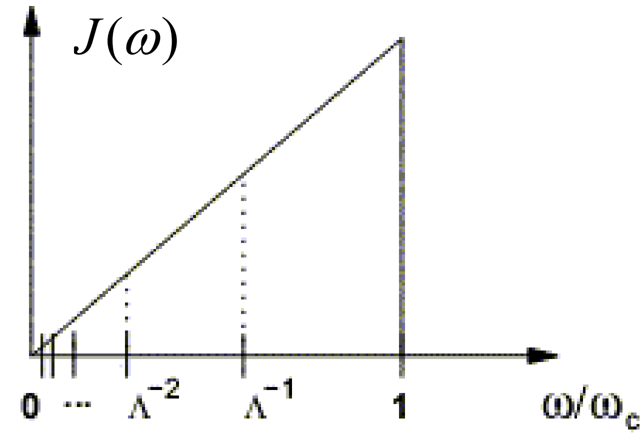
Bulla et al. (2005)

- Define logarithmic bins.
- Retain one bath state in each bin:

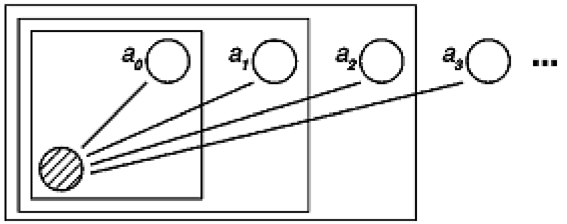
$$a_m = A_m^{-1} \int_{\text{bin } m} d\omega \sqrt{B(\omega)}\, a_\omega \,,$$

so that

$$H_{\text{host}} \approx \sum_{m=0}^{\infty} \omega_m\, a_m^\dagger\, a_m \,.$$



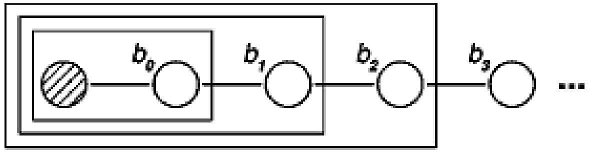
- The impurity interacts with all bins in a "**star**" configuration.



- Lanczos starting from

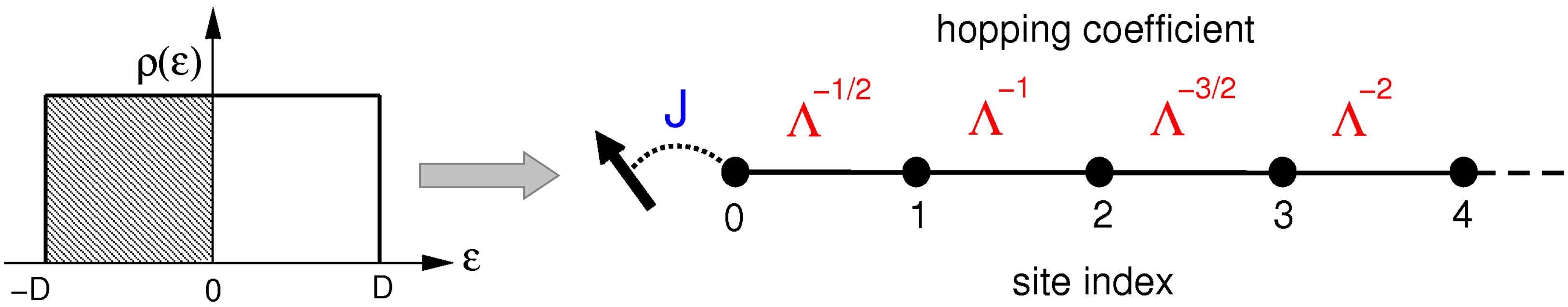
$$b_0 = B_0^{-1} \sum_{m=0}^{\infty} A_m\, a_m$$

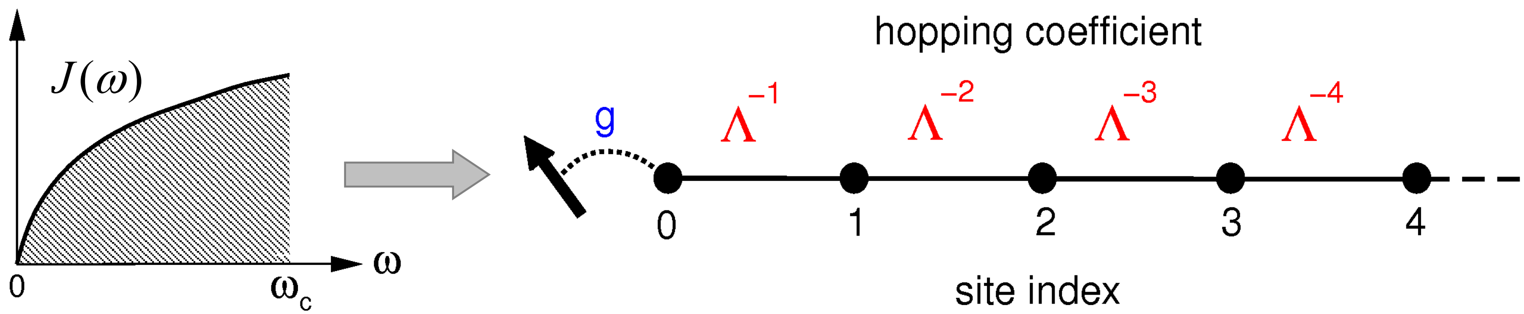
maps to a "**chain**" configuration.



Separation of energy scales

- Chain coefficients decay faster than for band electrons:





⇒ Discretization achieves the desired energy separation.

- Iterative solution of chains of length $L = 1, 2, 3, \ldots.$ should work if can restrict each chain site to just N_b basis states.

Does bosonic chain-NRG work?

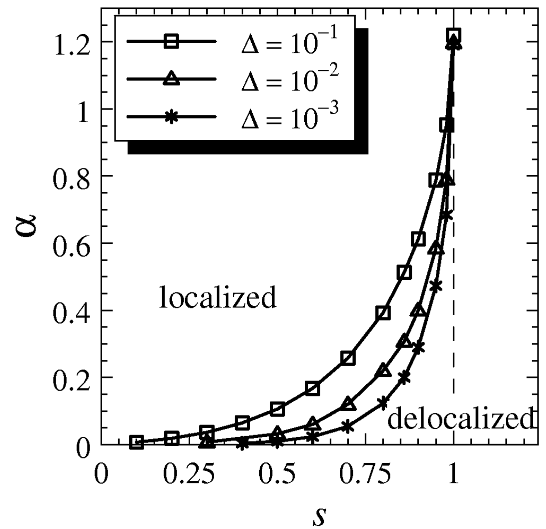
- The method has been tested very carefully on the spin-boson model [Bulla et al. (2003, 2005)]:

$$H = \sum_q \omega_q a_q^\dagger a_q - \Delta S_x + S_z N_q^{-1/2} \sum_q \lambda_q \left(a_q + a_q^\dagger \right).$$

with bath $J(\omega) = 2\pi\, \alpha\, \omega_c \left(\omega/\omega_c\right)^s$ for $0 < \omega < \omega_c$.

using a basis of the N_b lowest eigenstates of $b_n^\dagger b_n$.

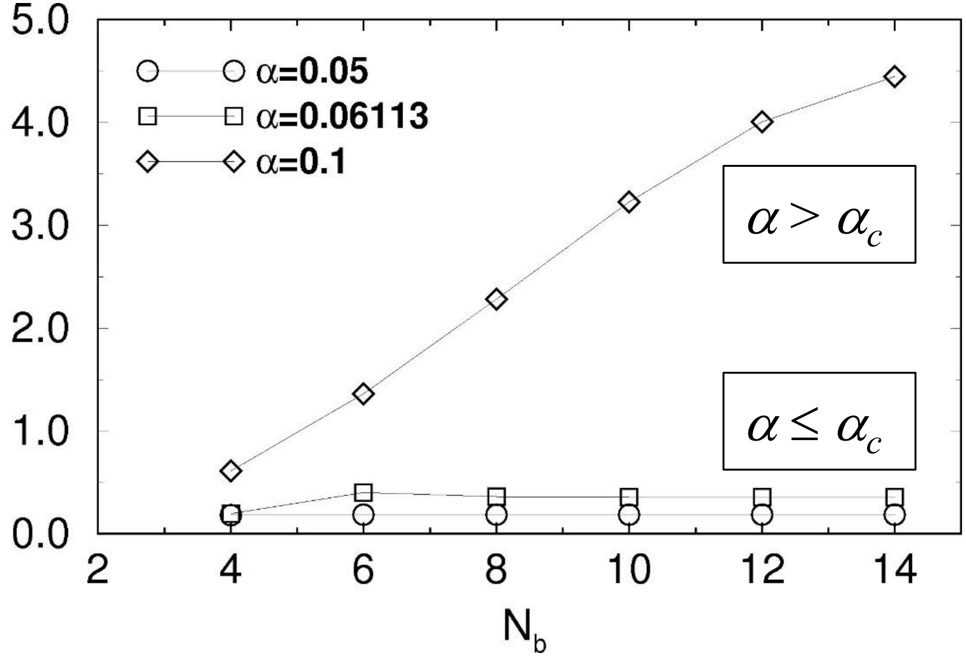
- As expected from other approaches,
  - for $0 < s < 1$, there is a quantum-critical point at $\alpha = \alpha_c(\Delta)$
  - for $s = 1$, instead have a Kosterlitz-Thouless quantum phase transition.



Does bosonic chain-NRG work?

- In the delocalized phase and at the critical point ($\alpha \leq \alpha_c$), results are stable and insensitive to choice of N_b.

$\langle b_{21}^\dagger b_{21} \rangle$



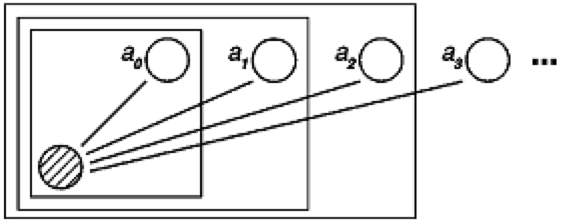
- In the localized phase ($\alpha > \alpha_c$) with $s < 1$, $\langle b_n^\dagger b_n \rangle$ diverges for large n, and bosonic chain-NRG fails.

Bosonic star-NRG

Bulla et al. (2005)

- In the limit $\Delta = 0$, S_z provides a static potential for the bosons. Under the star formulation,

$$H = \sum_m \omega_m a_m^\dagger a_m + S_z \sum_m A_m \left(a_m + a_m^\dagger \right)$$



with $\omega_m \propto \Lambda^{-m}, \;\; A_m \propto \Lambda^{-(1+s)m/2}$

- Can transform to displaced oscillators (for $\sigma = \pm 1$)

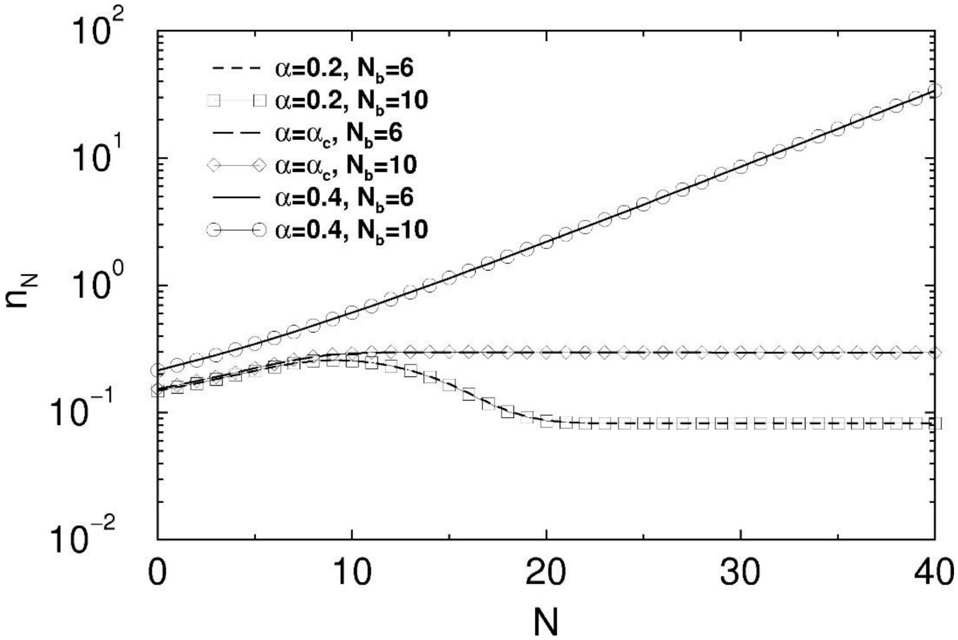
$$a_{m\sigma} = a_m + \sigma \, A_m \Big/ \left(2\sqrt{\pi} \omega_m \right)$$

such that $$H = \sum_{\sigma=\pm 1} \left| \sigma/2 \right\rangle \left\langle \sigma/2 \right| \sum_{m=0}^{\infty} \omega_m a_{m\sigma}^\dagger a_{m\sigma}$$

- Then $\left\langle a_m^\dagger a_m \right\rangle \approx \dfrac{A_m}{4\pi\omega_m^2} \sim \Lambda^{(1-s)m/2}$ diverges for $s < 1$.

Bosonic star-NRG

- For $\Delta \neq 0$, don't know exact oscillator shifts, but can use variational optimization of basis, then work with small N_b.



Translated back to chain language, $\langle b_n^\dagger b_n \rangle$ converges rapidly with increasing N_b.

- Main conclusion: bosonic-star NRG works well in localized phase, but not in delocalized phase.

Bosonic NRG: Successes and limitations

- With reasonable computational effort, bosonic NRG yields thermodynamics, dynamics, phase boundaries that are well-converged w.r.t. boson basis per site N_b and number of retained states N_s.
- Results are non-perturbative: not limited to small values of any model parameter.
- However, choice of bosonic basis is crucial:
  - Proper basis depends on location in phase diagram.
  - NRG described here is inefficient at basis optimization.
  - Variational product state NRG (Weichselbaum et al.) has advantages here.
- Critical behavior remains challenging (see lecture notes).

III: Bose-Fermi NRG

- Bose-Fermi Kondo model describes a spin-half $\mathbf{S}$ coupled to a conduction band **and** to 1-3 dissipative baths.
- Isotropic model has the Hamiltonian

$$H = \underbrace{J\mathbf{S}\cdot\mathbf{s} + H_{\text{band}}}_{H_{\text{Kondo}}} + \underbrace{g\mathbf{S}\cdot\mathbf{u} + H_{\text{bath}}}_{H_{\text{spin-boson}}}$$

where (for $\alpha = x, y, z$)

$$s_\alpha = \tfrac{1}{2}\sum\nolimits_{\sigma,\sigma'} c_{0\sigma}^\dagger \sigma_{\sigma\sigma'}^\alpha c_{0\sigma'} \qquad u_\alpha = a_{0\alpha} + a_{0\alpha}^\dagger$$

$$H_{\text{band}} = \sum\nolimits_{k,\sigma} \varepsilon_k \, c_{k\sigma}^\dagger \, c_{k\sigma} \qquad H_{\text{bath}} = \sum\nolimits_{q,\alpha} \omega_q \, a_{q\alpha}^\dagger \, a_{q\alpha}$$

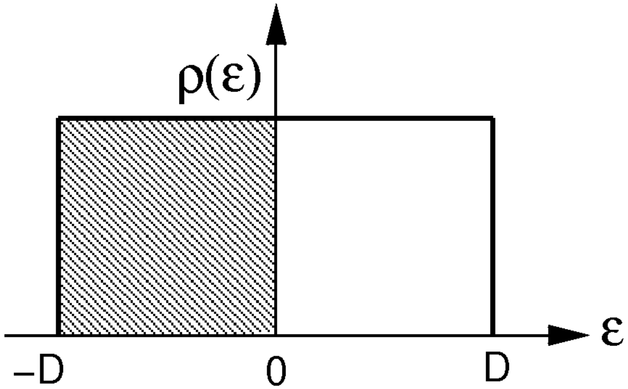
- Most studies have focused on one-bath case:

$$g\mathbf{S}\cdot\mathbf{u} \rightarrow gS_z u_z$$

Bose-Fermi Kondo model: Bath spectra

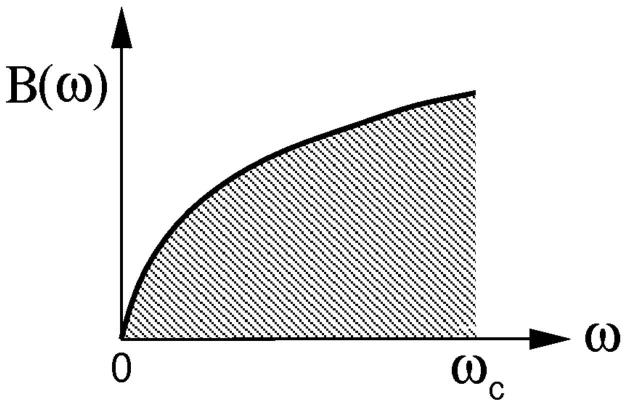
- Take a flat conduction band density of states:

$$\rho(\varepsilon) = \rho_0 \quad \text{for} \quad |\varepsilon| < D$$



- Assume a power-law bosonic spectrum:

$$B(\omega) = K_0^2\, \omega_c (\omega / \omega_c)^s$$



- Dimensionless couplings:

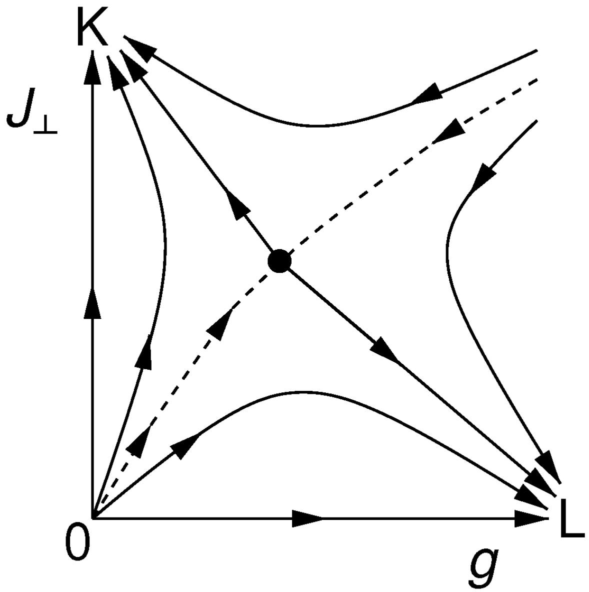
$$\rho_0 J \qquad\qquad K_0 g$$

One-bath Bose-Fermi Kondo model

For any sub-Ohmic bath exponent $0 < s < 1$, one-bath BFK model

$$H = J\mathbf{S}\cdot\mathbf{s} + H_{\text{band}} + gS_z u_z + H_{\text{bath}}$$

has a nontrivial critical point governing the boundary between Kondo and localized phases:

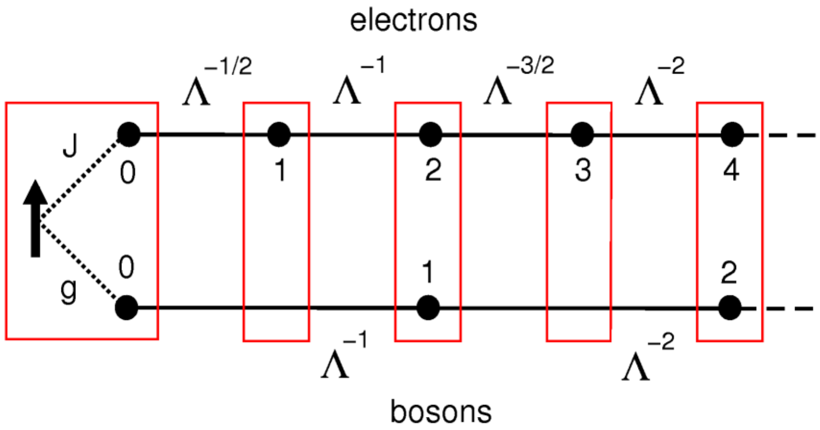


Embodies critical destruction of the Kondo effect, c.f. heavy fermions.

Bose-Fermi NRG

Glossop and KI (2005)

- Seek an NRG that treats simultaneously fermionic and bosonic degrees of freedom of the same energy.
- Guided by spin-boson model, use the chain NRG.
- Slightly complication: different Λ dependences of fermionic and bosonic tight-binding coefficients.
  - Could use different discretizations, $\Lambda_{\text{fermions}} = \Lambda^2_{\text{bosons}}$.
  - Instead, we add a bosonic site at every other iteration:

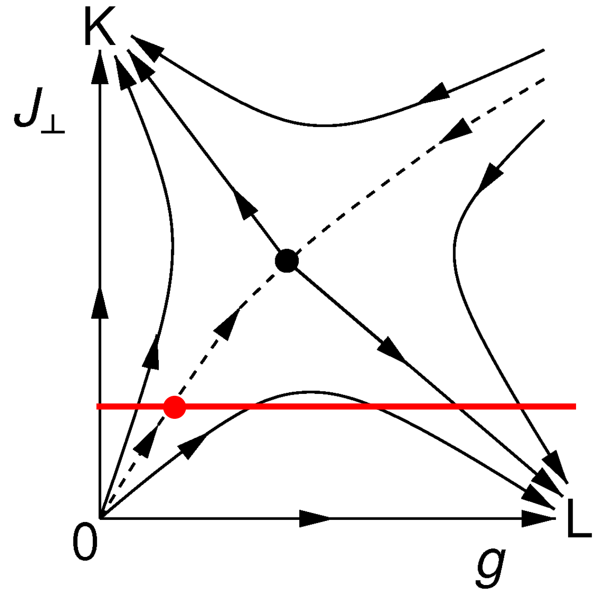


Testing the Bose-Fermi NRG

- The Ising-symmetry Bose-Fermi Kondo model is a good test ground: bosonization of the fermions maps problem onto the spin-boson model with an asymptotic bath spectrum

$$B(\omega) \propto \omega^{\min(1,\, s)}.$$

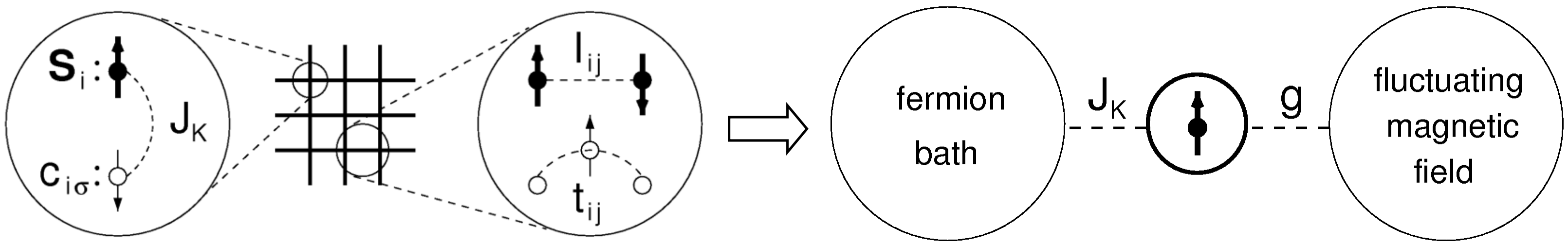
QPT should lie in the $s_{\text{spin-boson}} = \min(1, s)$ universality class (studied via bosonic NRG).



- BF-NRG reproduces spin-boson results, including their flaws.

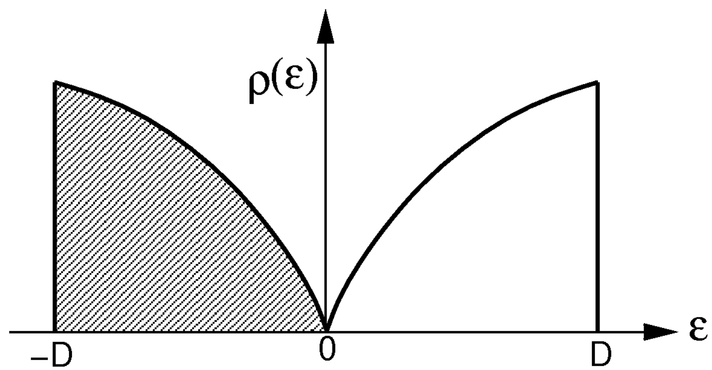
Bose-Fermi NRG beyond the BFK model

- Bose-Fermi NRG has been applied to a range of other problems:
  - Self-consistent Bose-Fermi Kondo model arising in the EDMFT treatment of the Kondo lattice.



Si et al. (2001)

  - Problems with a singular fermionic density of states as well as a sub-Ohmic bath (see lecture notes).



NRG with bosons: A scorecard

- The good: With appropriate choices of bosonic basis, NRG provides robust, non-perturbative solutions to a variety of interesting problems.
- The bad: NRG with bosons is not a “black-box” tool that can be applied indiscriminately, because the basis must be chosen appropriately for the regime of interest. May be fundamental problems near some critical points.
- The computationally ugly: With bosons, the basis grows very rapidly upon NRG iteration.

  Impedes extension to multi-impurity and multi-bath models, may be problematic for time-dependent studies.
- Key direction for future work: optimization of the bosonic basis.