Eliashberg Theory

AUTUMN SCHOOL ON CORRELATED ELECTRONS:
Emergent Phenomena in Correlated Matter

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BCS THEORY

$$\Delta(T) = -N(0)V \sum_{k'}^{\varepsilon_{k'}<\omega_D} \frac{\Delta(T)}{\sqrt{\varepsilon_{k'}{}^2 + \Delta^2(T)}} \tanh \frac{\sqrt{\varepsilon_{k'}{}^2 + \Delta^2(T)}}{2T}$$

$$V_{k,k'} \approx -\mathrm{V}\,\theta(\omega_D - \varepsilon_k)\theta(\omega_D - \varepsilon_{k'})$$

Two input parameters: the representative energy of bosonic spectra ω_D and the electron-boson coupling N(0)V. (ω_D can be obtained by experimental data)

Output: Δ(T) and Tc

$$\Delta(T=0) = 2\omega_D \exp[-1/N(0)V]$$

$$T_c = 1.13\omega_D \exp[-1/N(0)V]$$

$$2\Delta/T_c = 3.53 \quad \Delta C(T_c)/\gamma T_c = 1.43$$

The theory is approximatevily correct

only in weak coupling λ≈N(0)V<0.6

for λ > 0.6 the theory is correct

only in a qualitative way and not always

Range of application of the BCS theory

	λ	T_c (K)	$2\Delta/k_BT_c$	$\Delta C(T_c)/\gamma T_c$
Al	0.43	1.18	3.535	1.43
In	0.81	3.40	3.791	1.79
Pb	1.55	7.22	4.497	2.77
Nb_3Sn	1.7	17.90	4.567	2.64
Bi	2.45	6.11	4.916	2.03
$Pb_{0.75}Bi_{0.25}$	2. 76	6.91	5.119	2.27

Eliashberg theory

Existence of Fermi surface

$\lambda\omega_D/E_F$ <<1, first order Feynman diagrams in the Dyson equation (Migdal theorem)

Input parameters of Eliashberg theory

λ

Bosonic spectral function $\alpha^2(\omega)F(\omega)$

$$\lambda = 2\int_0^{+\infty} d\omega \frac{\alpha^2 F(\omega)}{\omega} \qquad \omega_{\log} = \exp\left[\frac{2}{\lambda}\int_0^{+\infty} d\omega \log(\omega)\frac{\alpha^2 F(\omega)}{\omega}\right]$$

Coulomb pseudopotential $\mu^*(\omega_c)$

$$
\begin{aligned}
H &= \sum_{\mathbf{k}} \epsilon_{\mathbf{k}} \psi^{\dagger}_{\mathbf{k}} \sigma_3 \psi_{\mathbf{k}} + \sum_{\mathbf{q}\nu} \omega_{\mathbf{q}\nu} b^{\dagger}_{-\mathbf{q}\nu} b_{\mathbf{q}\nu} + \sum_{\mathbf{k}\mathbf{k}'\nu} g_{\mathbf{k}\mathbf{k}'\nu} \varphi_{(\mathbf{k}-\mathbf{k}')\nu} \psi^{\dagger}_{\mathbf{k}'} \sigma_3 \psi_{\mathbf{k}} \\
&+ \frac{1}{2} \sum_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3\mathbf{k}_4} \langle \mathbf{k}_3\mathbf{k}_4 \,|V_c|\, \mathbf{k}_1\mathbf{k}_2 \rangle \, (\psi^{\dagger}_{\mathbf{k}_3} \sigma_3 \psi_{\mathbf{k}_1})(\psi^{\dagger}_{\mathbf{k}_4} \sigma_3 \psi_{\mathbf{k}_2})
\end{aligned}
$$

$$
\Psi_{\mathbf{k}} = \begin{pmatrix} c_{\mathbf{k}\uparrow} \\ c^{\dagger}_{-\mathbf{k}\downarrow} \end{pmatrix} \quad \varphi_{\mathbf{q}\nu} = b_{\mathbf{q}\nu} + b^{\dagger}{}_{-\mathbf{q}\nu}
$$

Nambu-Gorkov formalism

$$
\hat{G}(\mathbf{k}, i\omega_n) = \begin{bmatrix} G(\mathbf{k}, i\omega_n) & F(\mathbf{k}, i\omega_n) \\ F^{*}(\mathbf{k}, i\omega_n) & -G(-\mathbf{k}, -i\omega_n) \end{bmatrix}
$$

Dyson equation

$$
\hat{G}^{-1}(\mathbf{k}, i\omega_n) = \hat{G}_0^{-1}(\mathbf{k}, i\omega_n) - \hat{\Sigma}(\mathbf{k}, i\omega_n)
$$

$$\hat{\Sigma}(\mathbf{k}, i\omega_n) = \hat{\Sigma}_{\mathrm{ep}}(\mathbf{k}, i\omega_n) + \hat{\Sigma}_{\mathrm{c}}(\mathbf{k}, i\omega_n),$$

with

$$\hat{\Sigma}_{\mathrm{ep}}(\mathbf{k}, i\omega_n) = -T \sum_{\mathbf{k}'n'} \hat{\tau}_3 \hat{G}(\mathbf{k}', i\omega_{n'}) \hat{\tau}_3 \times \sum_{\lambda} |g_{\mathbf{k}\mathbf{k}'\nu}|^2 D_\nu(\mathbf{k}-\mathbf{k}', i\omega_n - i\omega_{n'}),$$

and

$$\hat{\Sigma}_{\mathrm{c}}(\mathbf{k}, i\omega_n) = -T \sum_{\mathbf{k}'n'} \hat{\tau}_3 \hat{G}^{\mathrm{od}}(\mathbf{k}', i\omega_{n'}) \hat{\tau}_3 V(\mathbf{k}-\mathbf{k}').$$

$D_\nu(\mathbf{q}, i\omega_n) = 2\omega_{\mathbf{q}\nu} / \left[(i\omega_n)^2 - \omega_{\mathbf{q}\nu}^2\right]$ is the dressed propagator for phonons with momentum $\mathbf{q}$ and branch index ν, and $g_{\mathbf{k}\mathbf{k}'\nu}$ is the screened electron-phonon matrix element for the scattering between the electronic states $\mathbf{k}$ and $\mathbf{k}'$ through a phonon with wavevector $\mathbf{q} = \mathbf{k}' - \mathbf{k}$, frequency $\omega_{\mathbf{q}\nu}$ and branch index ν

Migdal theorem: if $\lambda\omega_D/E_F$ <<1

only first order Feyman diagrams

$$\Sigma(\mathbf{k}, i\omega_n) = i\omega_n [1 - Z(\mathbf{k}, i\omega_n)] I + \chi(\mathbf{k}, i\omega_n)\sigma_3 + \phi(\mathbf{k}, i\omega_n)\sigma_1 + \bar{\phi}(\mathbf{k}, i\omega_n)\sigma_2,$$

$$G(\mathbf{k}, i\omega_n)^{-1} = i\omega_n Z I - (\epsilon_{\mathbf{k}} + \chi)\sigma_3 - \phi\sigma_1 - \bar{\phi}\sigma_2$$

$$\hat{G}(\mathbf{k}, i\omega_n) = -\{i\omega_n Z(\mathbf{k}, i\omega_n)\hat{\tau}_0 + [\epsilon_{\mathbf{k}} + \chi(\mathbf{k}, i\omega_n)]\hat{\tau}_3 + \phi(\mathbf{k}, i\omega_n)\hat{\tau}_1\} / \Theta(\mathbf{k}, i\omega_n), \quad (13)$$

where the denominator is defined as:

$$\Theta(\mathbf{k}, i\omega_n) = [\omega_n Z(\mathbf{k}, i\omega_n)]^2 + [\epsilon_{\mathbf{k}} + \chi(\mathbf{k}, i\omega_n)]^2 + [\phi(\mathbf{k}, i\omega_n)]^2 . \quad (14)$$

Now we impose self-consistency by replacing the esplicit expression for the Green functions inside the self energy expression.

$$[1 - Z(\mathbf{k}, i\omega_n)]\, i\omega_n = \frac{1}{\beta} \sum_{\mathbf{k}', n', \nu} |g_{\mathbf{k},\mathbf{k}',\nu}|^2 \frac{i\omega_{n'} Z(\mathbf{k}', i\omega_{n'}) D_\nu(\mathbf{k} - \mathbf{k}', i\omega_n - i\omega_{n'})}{\Theta(\mathbf{k}', i\omega_{n'})}$$

$$\chi(\mathbf{k}, i\omega_n) = \frac{1}{\beta} \sum_{\mathbf{k}', n', \nu} |g_{\mathbf{k},\mathbf{k}',\nu}|^2 \frac{\chi(\mathbf{k}', i\omega_{n'}) + \varepsilon_{\mathbf{k}'}}{\Theta(\mathbf{k}', i\omega_{n'})} D_\nu(\mathbf{k} - \mathbf{k}', i\omega_n - i\omega_{n'})$$

$$\phi(\mathbf{k}, i\omega_n) = -\frac{1}{\beta} \sum_{\mathbf{k}', n', \nu} [|g_{\mathbf{k},\mathbf{k}',\nu}|^2 D_\nu(\mathbf{k} - \mathbf{k}', i\omega_n - i\omega_{n'}) - V_C(\mathbf{k} - \mathbf{k}')] \frac{\phi(\mathbf{k}', i\omega_{n'})}{\Theta(\mathbf{k}', i\omega_{n'})}$$

$$\bar{\phi}(\mathbf{k}, i\omega_n) = -\frac{1}{\beta} \sum_{\mathbf{k}', n', \nu} [|g_{\mathbf{k},\mathbf{k}',\nu}|^2 D_\nu(\mathbf{k} - \mathbf{k}', i\omega_n - i\omega_{n'}) - V_C(\mathbf{k} - \mathbf{k}')] \frac{\bar{\phi}(\mathbf{k}', i\omega_{n'})}{\Theta(\mathbf{k}', i\omega_{n'})}$$

$$n = 1 - \frac{2}{\beta} \sum_{\mathbf{k}', n'} \frac{\chi(\mathbf{k}', i\omega_{n'}) + \varepsilon_{\mathbf{k}'} - \mu}{\Theta(\mathbf{k}', i\omega_{n'})}$$

$$\alpha^2 F(\mathbf{k}, \mathbf{k}', \omega) = N_{\mathrm{F}} \sum_{\nu} |g_{\mathbf{k}\mathbf{k}'\nu}|^2 \delta(\omega - \omega_{\mathbf{k}-\mathbf{k}',\nu}).$$

$$Z(\mathbf{k}, i\omega_n) \to \langle Z(\mathbf{k}, i\omega_n) \rangle_{\varepsilon = E_F} = Z(i\omega_n)$$

$$\phi(\mathbf{k}, i\omega_n) \to \langle \phi(\mathbf{k}, i\omega_n) \rangle_{\varepsilon = E_F} = \phi(i\omega_n)$$

$$\chi(\mathbf{k}, i\omega_n) \to \langle \chi(\mathbf{k}, i\omega_n) \rangle_{\varepsilon = E_F} = \chi(i\omega_n)$$

Including the repulsive term in the Eliashberg equations is a hard task. The Coulomb interaction cannot be introduced with the same accuracy of the electron-phonon one, since it does not have a natural cut-off to ensure a convergent sum on the Matsubara's frequencies.
The electron-electron interaction has a large energy scale (and then a narrow interaction time) with respect to electron-phonon attraction. The electron-phonon interaction has a timescale typical of the much larger inverse phonon frequencies. The time scale difference is normally dealt using an energy window ω_C with a renormalized electron-electron interaction

$$\mu^* = \frac{\mu}{1 + \mu \ln\left(E_F/\omega_C\right)}, \tag{42}$$

which is called *Morel-Anderson pseudopotential*. In this formula, μ is an average electron-electron matrix element times the density of states at the Fermi level.
In the normal state self-energy the Coulomb potential is included, therefore only the off-diagonal term will be affected by this correction

$$\mu^* \sim 0.1$$

Three different but equivalent formulations: imaginary, real and mixed

$$\Delta(i\omega_n)Z(i\omega_n) = \frac{\pi}{\beta}\sum_{\omega_{n'}}\frac{\Delta(i\omega_{n'})}{\sqrt{\omega^2_{n'}+\Delta^2(i\omega_n)}}\left[\lambda(i\omega_{n'}-i\omega_n)-\mu^*(\omega_C)\right]\theta(\omega_C-|\omega_{n'}|)$$

$$Z(i\omega_n) = 1+\frac{\pi}{\omega_n\beta}\sum_{\omega_{n'}}\frac{\omega_{n'}}{\sqrt{\omega^2_{n'}+\Delta^2(i\omega_n)}}\lambda(i\omega_{n'}-i\omega_n) \qquad (41)$$

where $\lambda(i\omega_{n'}-i\omega_n)$ is a function related to the electron-boson spectral density $\alpha^2F(\omega)$ through the relation

$$\lambda(i\omega_n-i\omega_n) = 2\int_0^\infty \frac{\Omega\alpha^2F(\Omega)d\Omega}{\Omega^2+(\omega_{n'}-\omega_n)^2}. \qquad \mu^* = \frac{\mu}{1+\mu\ln\left(E_F/\omega_C\right)}$$

Iterative solution

Determination of T_c

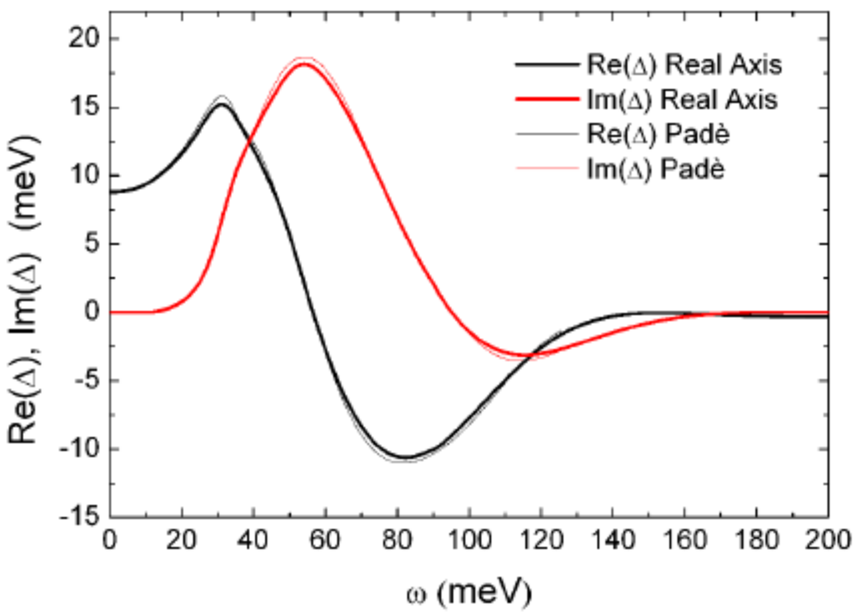
$\Delta=\Delta(i\omega_{n=0})$ is the value of the superconductive gap only in weak coupling regime: $\lambda<<1$

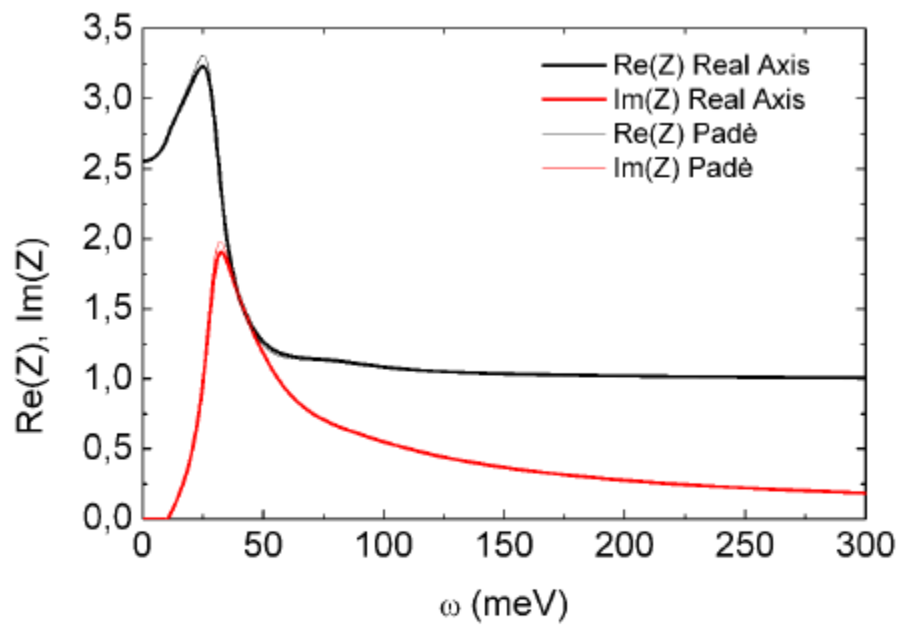
Padè approximants $T<T_c/10$

Padé approximants method

$$T < T_C/10$$

Lead





If there are magnetic impurities this method don't work (of course always T<<Tc)

$$
\begin{aligned}
\Delta(\omega,T)Z(\omega,T) = & \int_0^{\omega_C} d\omega' \,\mathrm{Re}\left[\frac{\Delta(\omega',T)}{\sqrt{\omega' - \Delta^2(\omega',T)}}\right] \int_0^{\infty} d\Omega \alpha^2 F(\Omega) \\
& \times \left\{ [n(\Omega) + f(-\omega')] \left[\frac{1}{\omega + \omega' + \Omega + i\delta^+} - \frac{1}{\omega - \omega' - \Omega + i\delta^+}\right] \right. \\
& \left. - [n(\Omega) + f(\omega')] \left[\frac{1}{\omega - \omega' + \Omega + i\delta^+} - \frac{1}{\omega + \omega' - \Omega + i\delta^+}\right] \right\} \\
& - \mu^* \int_0^{\omega_C} d\omega' \mathrm{Re}\left[\frac{\Delta(\omega',T)}{\sqrt{\omega' - \Delta^2(\omega',T)}}\right] [1 - 2f(\omega')] \,, \\
[1 - Z(\omega,T)]\,\omega = & \int_0^{\infty} d\omega' \,\mathrm{Re}\left[\frac{\omega'}{\sqrt{\omega' - \Delta^2(\omega',T)}}\right] \int_0^{\infty} d\Omega \alpha^2 F(\Omega) \\
& \times \left\{ [n(\Omega) + f(-\omega')] \left[\frac{1}{\omega + \omega' + \Omega + i\delta^+} - \frac{1}{\omega - \omega' - \Omega + i\delta^+}\right] \right. \\
& \left. - [n(\Omega) + f(\omega')] \left[\frac{1}{\omega - \omega' + \Omega + i\delta^+} - \frac{1}{\omega + \omega' - \Omega + i\delta^+}\right] \right\} .
\end{aligned}
$$

$$
\begin{aligned}
\tilde{\omega}(\omega) &= \omega + i\pi T \sum_{m=1}^{\infty} \frac{\tilde{\omega}(i\omega_m)}{\sqrt{\tilde{\omega}^2(i\omega_m) + \phi^2(i\omega_m)}} \left[\lambda(\omega - i\omega_m) - \lambda(\omega + i\omega_m)\right] \\
&\quad + i\pi \int_{-\infty}^{\infty} dz \frac{\tilde{\omega}(\omega - z)}{\sqrt{\tilde{\omega}^2(\omega - z) - \phi^2(\omega - z)}} \alpha^2 F(z) \left[n(z) + f(z - \omega)\right]
\end{aligned}
$$

$$
\begin{aligned}
\phi(\omega) &= i\pi T \sum_{m=1}^{\infty} \frac{\phi(i\omega_m)}{\sqrt{\tilde{\omega}^2(i\omega_m) + \phi^2(i\omega_m)}} \left[\lambda(\omega - i\omega_m) - \lambda(\omega + i\omega_m) - 2\mu^* \theta(\omega_C - |\omega_m|)\right] \\
&\quad + i\pi \int_{-\infty}^{\infty} dz \frac{\phi(\omega - z)}{\sqrt{\tilde{\omega}^2(\omega - z) - \phi^2(\omega - z)}} \alpha^2 F(z) \left[n(z) + f(z - \omega)\right].
\end{aligned}
$$

Input parameters: $\alpha^2F(\omega)$ and μ^*

Numerical solution of Eliashberg Equations

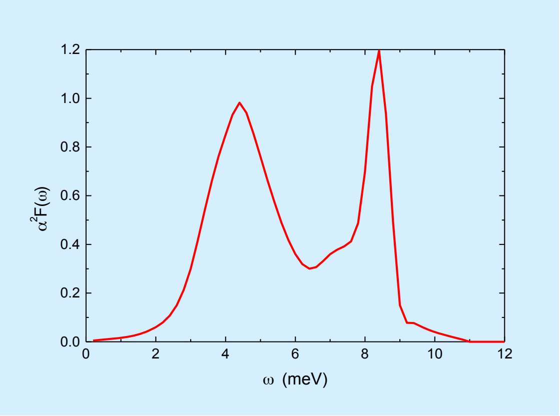
Complex function $\Delta(\omega,T)$ and $Z(\omega,T)$

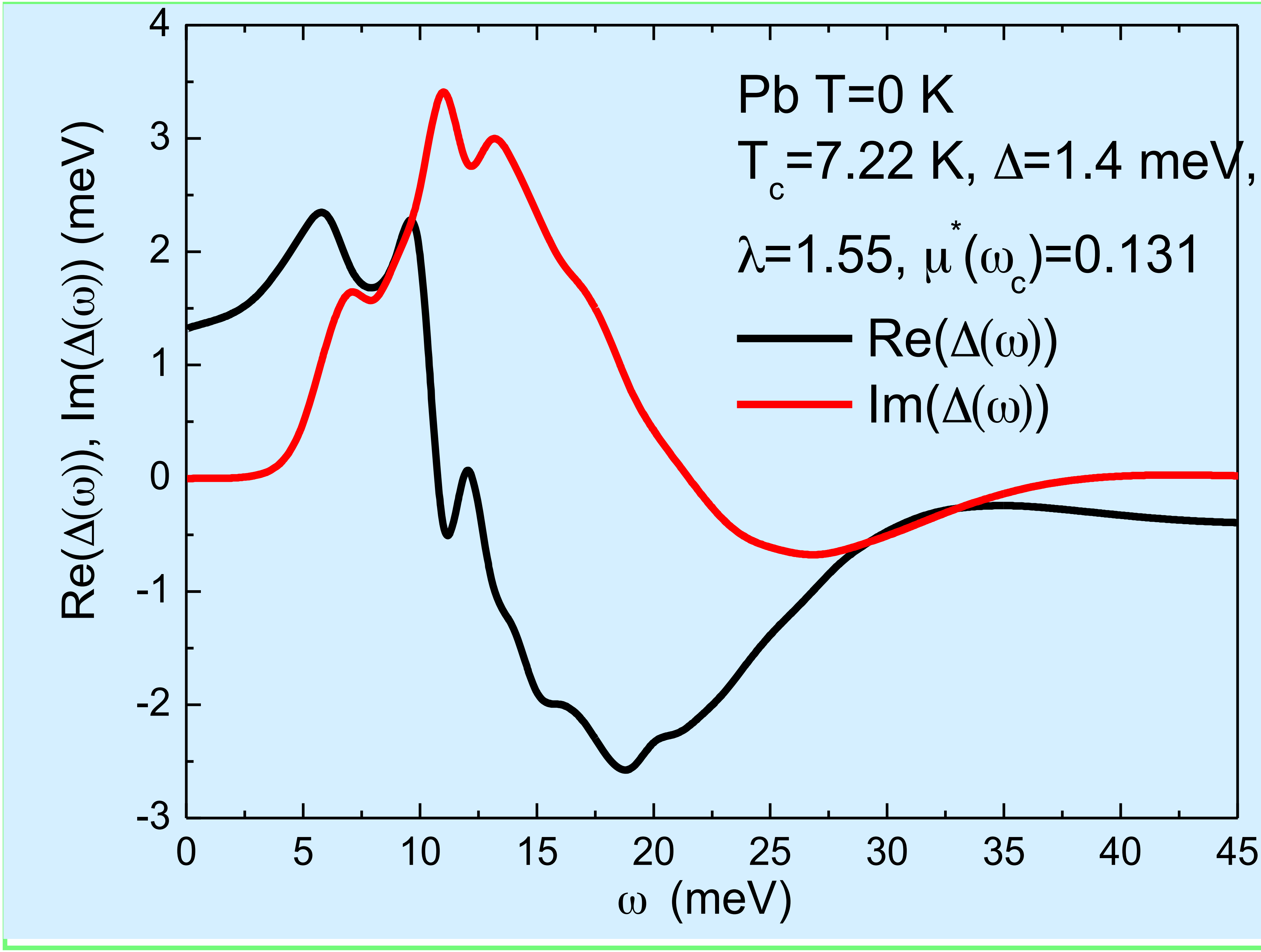
All superconductive physical observables

are simple functions of $\Delta(\omega,T)$ and $Z(\omega,T)$

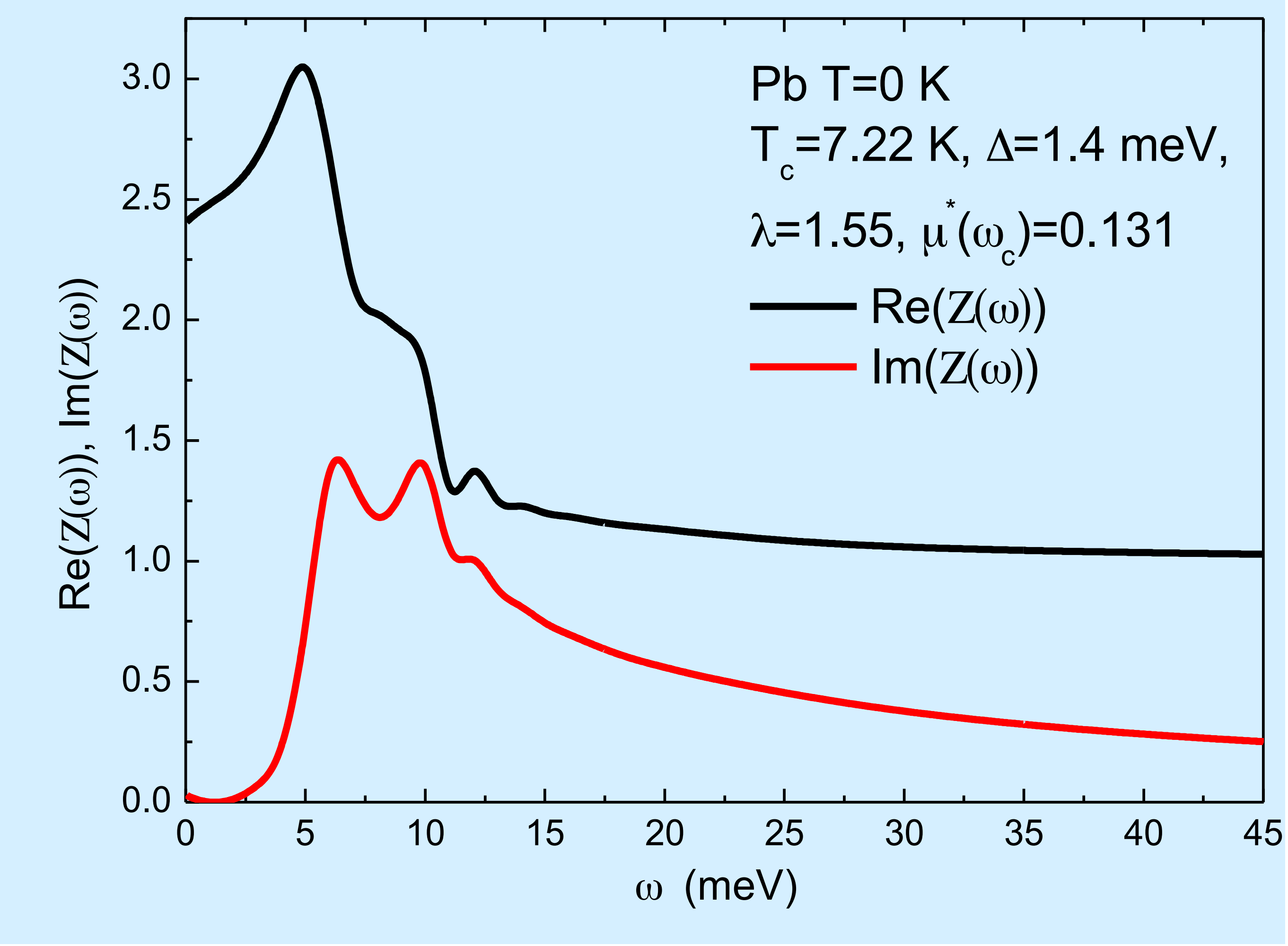
in BCS theory $\Delta(\omega,T)$ is a number

and $Z(\omega,T)=1$





Pb T=0 K
T_c=7.22 K, Δ=1.4 meV,
λ=1.55, μ*(ω_c)=0.131
Re(Δ(ω))
Im(Δ(ω))
Re(Δ(ω)), Im(Δ(ω)) (meV)
ω (meV)
4
3
2
1
0
-1
-2
-3
0
5
10
15
20
25
30
35
40
45



Pb T=0 K
T$_c$=7.22 K, Δ=1.4 meV,
λ=1.55, μ*(ω$_c$)=0.131
Re(Z(ω))
Im(Z(ω))
Re(Z(ω)), Im(Z(ω))
ω (meV)
0.0
0.5
1.0
1.5
2.0
2.5
3.0
0
5
10
15
20
25
30
35
40
45

BCS limit

In order to better understand these equations, it can be useful to reduce them to BCS limit. To achieve this aim further approximations are introduced. First of all, all bosons factor in the real-axis Eliashberg equations are ignored, i.e., real bosons scattering are not taken into account. Further, the imaginary parts of Δ and Z must be neglected and one can set $\Delta(\omega, T) = \Delta_0(T)$ for $\omega' < \omega_D$ and $\Delta(\omega, T) = 0$ for $\omega' \geq \omega_D$ where $\Delta_0(T)$ is a real number and ω_D is the Deybe energy. And $Z(\omega, T)$ can be replaced by its value in the normal state at $\omega = 0$ and $T = 0$, then

$$Z(0,T) - 1 = 2\int_0^{\infty} d\omega' \int_0^{\infty} d\Omega\, \alpha^2 F(\Omega) \left[\frac{f(-\omega')}{(\omega'+\Omega)^2} + \frac{f(\omega')}{(\omega'+\Omega)^2}\right] \equiv \lambda(T) \tag{45}$$

and, in the $T \to 0$ limit it becomes

$$Z(0,0) - 1 = \int_0^{\infty} d\Omega\, \alpha^2 F(\Omega) \int_0^{\infty} \frac{2d\omega'}{(\omega'+\Omega)^2} \equiv \lambda. \tag{46}$$

The gap equation becomes

$$\Delta_0(T) = \int_{\Delta_0(T)}^{\omega_D} d\omega' \frac{\Delta_0(T)}{\sqrt{\omega'^2 - \Delta^2{}_0(T)}} \frac{\lambda - \mu^*}{1+\lambda} \Big[1 - 2f(\omega')\Big]. \tag{47}$$

It is interesting to note that now ω_D is important for both λ and μ contribution.
If one consider $\varepsilon = \sqrt{\omega'^2 - \Delta^2{}_0}$, the equation can be rewritten as

$$\Delta_0(T) = \frac{\lambda - \mu^*}{1+\lambda} \int_0^{\omega_D} d\varepsilon \frac{\Delta_0(T)}{\sqrt{\varepsilon^2 + \Delta^2{}_0(T)}} \left[1 - 2f\left(\sqrt{\varepsilon^2 + \Delta^2{}_0(T)}\right)\right], \tag{48}$$

which is the usual BCS equation at finite temperature. In the the $T \to 0$ limit

$$\Delta_0 = \frac{\lambda - \mu^*}{1+\lambda} \int_0^{\omega_C} d\varepsilon \frac{\Delta_0}{\sqrt{\varepsilon^2 + \Delta^2{}_0}}\ , \tag{49}$$

Tunneling: junction S I N

$$I_S(V) \propto \int d\omega \Re \left[\frac{|\omega|}{\sqrt{\omega^2 - \Delta^2(\omega)}} \right] [f(\omega) - f(\omega + V)]$$

$$\left(\frac{dI}{dV}\right)_S \Big/ \left(\frac{dI}{dV}\right)_N \propto \Re \left\{ \frac{|V|}{\sqrt{V^2 - \Delta^2(V)}} \right\}$$

at very low T the current derivative is proportional at the superconductive density of states

$\mathrm{Re}[\Delta(\omega, T)] = \omega$ **the superconductive gap**

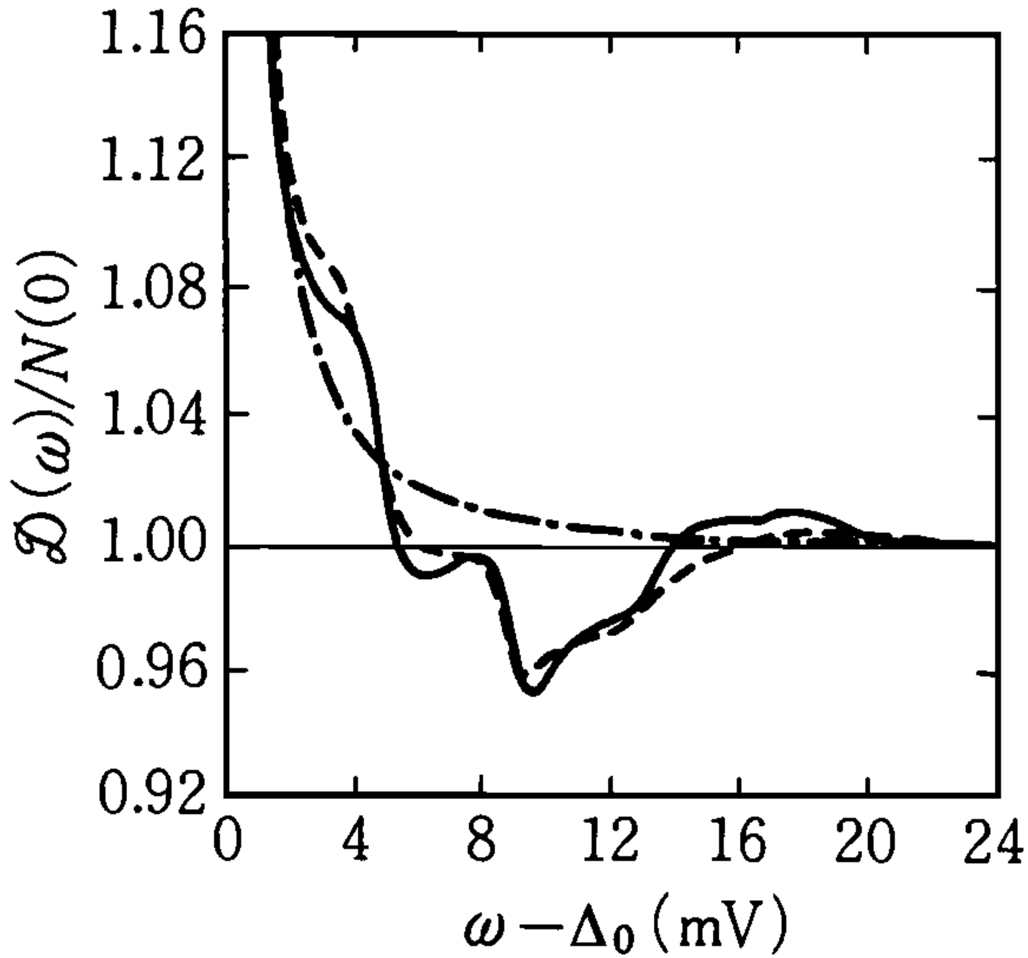


Fig. 4.15. The energy dependence of the density of states of lead in a superconducting state. The broken line represents the data obtained from the tunnelling effect while the BCS weak-coupling theory yields the dot-and-broken line. The strong-coupling result is shown by a solid line. (J.R. Schrieffer, *et al.*: [C-3].)

Real axis Eliashberg Equations at T=0

$$\Delta(\omega)\cdot Z(\omega) = \int_0^{+\infty} d\omega' \cdot n_1(\omega')\cdot\left[K_+(\omega,\omega') - \mu^*\right]\theta(\omega_c - \omega')$$

$$\left[1 - Z(\omega)\right]\cdot\omega = \int_0^{+\infty} d\omega' \cdot n_2(\omega')\cdot K_-(\omega,\omega')$$

$$K_\pm(\omega,\omega') = \int_0^{+\infty} d\Omega\cdot\alpha^2(\Omega)F(\Omega)\cdot\left[\frac{1}{\omega'+\omega+\Omega+i\delta} \pm \frac{1}{\omega'-\omega+\Omega-i\delta}\right]$$

$$n_1(\omega') = \mathrm{Re}\left\{\frac{\Delta(\omega')}{\sqrt{\omega'^2 - \Delta^2(\omega')}}\right\}$$

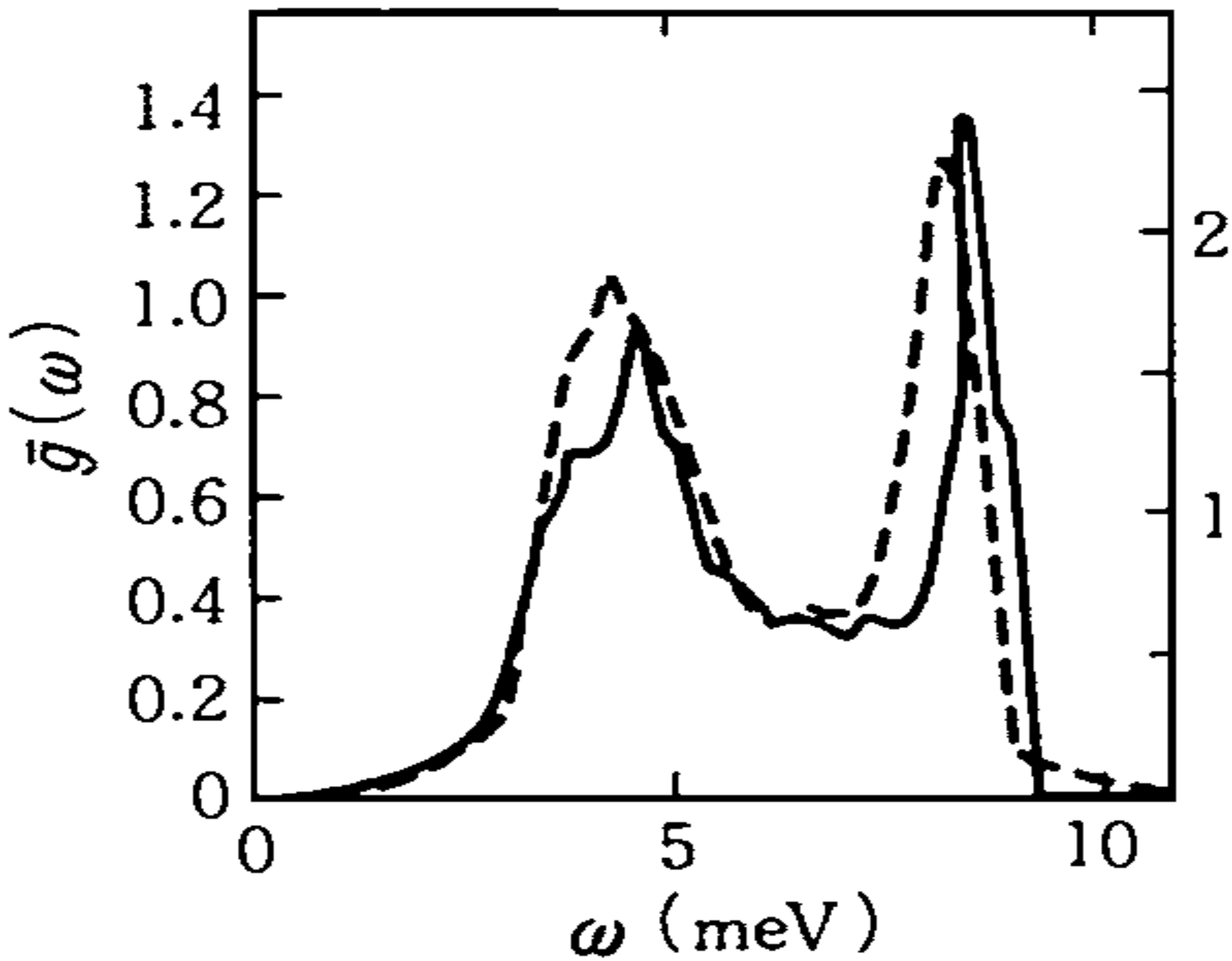
$$n_2(\omega') = \mathrm{Re}\left\{\frac{\omega'}{\sqrt{\omega'^2 - \Delta^2(\omega')}}\right\}$$

The procedure followed is conceptually simple but mathematically complex [20,21]. A first guess is made for the two quantities, namely $\alpha^2F_0(\omega)$, i.e. starting with a generic function greater than zero in a finite range and with a Coulomb pseudopotential parameter, $\mu_0^* \simeq 0.1$. So the Eliashberg equations (at $T = 0$) can be solved numerically with these two input parameters in order to obtain the complex function $\Delta(\omega)$ necessary for calculating the superconductive density of states $N_c^0(\omega)$ denoted by the subscript c (calculated) and 0 (for a first choice).

Moreover the functional derivative $\frac{\delta N_c^0(\omega)}{\delta\alpha^2F(\nu)}$ which give the infinitesimal response of $N_c^0(\omega)$ to chance $\alpha^2F(\nu)$ is computed. This is used to make a second guess for $\alpha^2F(\nu)$ through the equation $\delta\alpha^2F_0(\nu) = \int d\omega[\frac{\delta N_c^0(\omega)}{\delta\alpha^2F(\nu)}]^{-1}[N_m(\omega) - N_c^0(\omega)]$. The new electron phonon spectral function is $\alpha^2F_1(\nu) = \alpha^2F_0(\nu) + \delta\alpha^2F_0(\nu)$. This procedure is continued until convergence is reached. An

$$\mathrm{Re}[\Delta(\omega)] = \omega$$

$$\Delta_c = \Delta_{\exp} \longrightarrow \mu^*$$



1.4
1.2
1.0
0.8
0.6
0.4
0.2
0
$\bar{g}(\omega)$
2
1
0
5
10
ω (meV)

In the past $\alpha^2F(\omega)$ and μ^*

had obtained by experimental data

(also for no superconducting materials

via the proximity effect)

or was free parameters

but now can be calculated by

DENSITY FUNCTIONAL THEORY (DFT)

No free parameters !

Physical observables can be calculated:

Critical temperature

Gap value as function of temperature

Superconductive density of states

Upper critical field as function of temperature

Specific heat as function of temperature

NMR

Magnetic susceptivity

Isotopic effect

Etc…

T_c Equations

BCS λ<<1 analityc solution

$$T_C = 1.13\omega_D \exp\left[-\frac{1+\lambda}{\lambda-\mu^*}\right]$$

Eliashberg λ<1.5 μ*<0.15 num solution

$$T_C = \frac{\omega_D}{1.2}\exp\left[-\frac{1.04(1+\lambda)}{\lambda-\mu^*(1+0.62\lambda)}\right]$$

Eliashberg λ>10 analityc solution

$$T_C \approx 0.183\omega_D\sqrt{\lambda}$$

Standard Eliashberg Equations

three different but equivalent formulations:

real axis, imaginary axis and mixed

Approximations of Eliashberg Theory, in red the possible generalizations

Validity of Migdal Theorem $\lambda\omega_D/E_F << 1$, HTCS, fullerenes

Spin of Cooper pair =0 Sr_2RuO_4

Isotropic order parameter $\Delta(\omega)$ HTCS

Conduction bandwidth infinite Fullerenes, HTCS

Half filling of conduction band HTCS

One conduction band MgB2, iron pnictides

Normal density of state constant around the Fermi level Nb_3Sn, $PuCoGa_5$

No effect of disorder and magnetic impurities all superconductors

Breakdown of Migdal theorem

corrections at the first order in $\lambda\omega_D/E_F$ at T=Tc

$$Z(i\omega_n)\Delta(i\omega_n)=\pi T_c\sum_{\omega_m}\frac{\omega_0^2}{(\omega_n-\omega_m)^2+\omega_0^2}\frac{\Delta(i\omega_m)}{|\omega_m|}\lambda[1+2\lambda P_V(i\omega_n,i\omega_m;Q_c)+\lambda P_c(i\omega_n,i\omega_m;Q_c)]\times\frac{2}{\pi}\arctan\left[\frac{E}{2Z(i\omega_m)|\omega_m|}\right],$$

$$Z(i\omega_n)=1+\frac{\pi T_c}{\omega_n}\sum_{\omega_m}\frac{\omega_0^2}{(\omega_n-\omega_m)^2+\omega_0^2}\lambda[1+\lambda P_V(i\omega_n,i\omega_m;Q_c)]\frac{\omega_m}{|\omega_m|}\frac{2}{\pi}\arctan\left[\frac{E}{2Z(i\omega_m)|\omega_m|}\right].$$

$$|g_{\mathbf{p},\mathbf{k}}|^2=g^2\left[\frac{2k_F}{q_c}\right]^2\theta(q_c-|\mathbf{p}-\mathbf{k}|)$$

$$m=\omega_0/E_F$$

L. Pietronero, S. Str¨assler and C. Grimaldi, Phys. Rev. B 52, 10516 (1995)

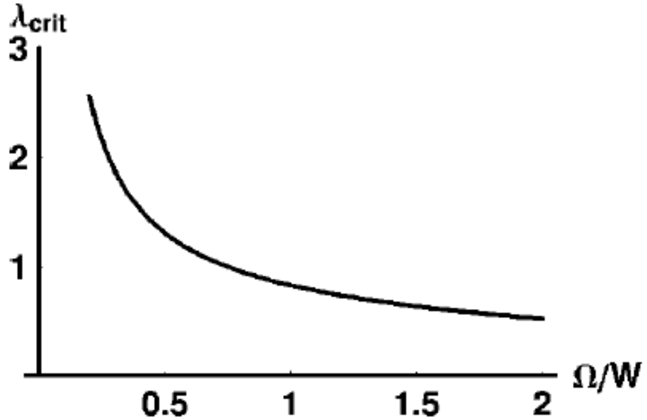
C. Grimaldi, L. Pietronero and S. Str¨assler, Phys. Rev. B 52, 10530 (1995)

$$T_c=\frac{1.13\omega_0}{\sqrt{e}\,(1+m)}\exp\left[\frac{1}{2}\frac{m}{(1+m)}\right]\exp\left[-\frac{1+\lambda_z[1/(1+m)]}{\lambda_\Delta}\right]$$

$$\lambda_\Delta=\lambda_\Delta(\lambda,m,Q_c)=\lambda[1+2\lambda P_V(m,Q_c)+\lambda P_c(m,Q_c)]$$

$$\lambda_z=\lambda_z(\lambda,m,Q_c)=\lambda[1+\lambda P_V(m,Q_c)]$$

A λeff >> λ in the standard Eliashberg equations can reproduce the vertex corrections



Stable system ⇒ $Z(i\omega_n)>1$ (λz>0)

O.V. Danylenko, O.V. Dolgov, PRB, VOLUME 63, 094506, (2001)

O.V. Dolgov and V.V. Losyakov, Phys. Lett. A 190, 189 (1994)

λ>>1 ? YES

COPPER OXIDES (overdoped), PuCoGa5 etc (d-wave)

$$\omega_n Z(i\omega_n,\phi)=\omega_n+\pi T\sum_m\int_0^{2\pi}\frac{d\phi'}{2\pi}\Lambda(i\omega_n-i\omega_m,\phi,\phi')N_Z(i\omega_m,\phi')+\Gamma\frac{N_Z(i\omega_n)}{c^2+N_Z(i\omega_n)^2} \qquad (60)$$

$$Z(i\omega_n,\phi)\Delta(i\omega_n,\phi)=\pi T\sum_m\int_0^{2\pi}\frac{d\phi'}{2\pi}[\Lambda(i\omega_n-i\omega_m,\phi,\phi')-\mu^*(\phi,\phi',\omega_c)\vartheta(\omega_c-\omega_m)]N_\Delta(i\omega_m,\phi') \qquad (61)$$

$$\Lambda(i\omega_n-i\omega_m,\phi,\phi')=2\int_0^{+\infty}d\Omega\alpha^2F(\Omega,\phi,\phi')/[(\omega_n-\omega_m)^2+\Omega^2],$$

$$N_\Delta(i\omega_m,\phi)=\frac{\Delta(i\omega_m,\phi)}{\sqrt{\omega_m^2+\Delta^2(i\omega_m,\phi)}},$$

$$N_Z(i\omega_m,\phi)=\frac{\omega_m}{\sqrt{\omega_m^2+\Delta^2(i\omega_m,\phi)}},$$

$$N_Z(i\omega_n)=\frac{1}{2\pi}\int_0^{2\pi}N_Z(i\omega_n,\phi)d\phi.$$

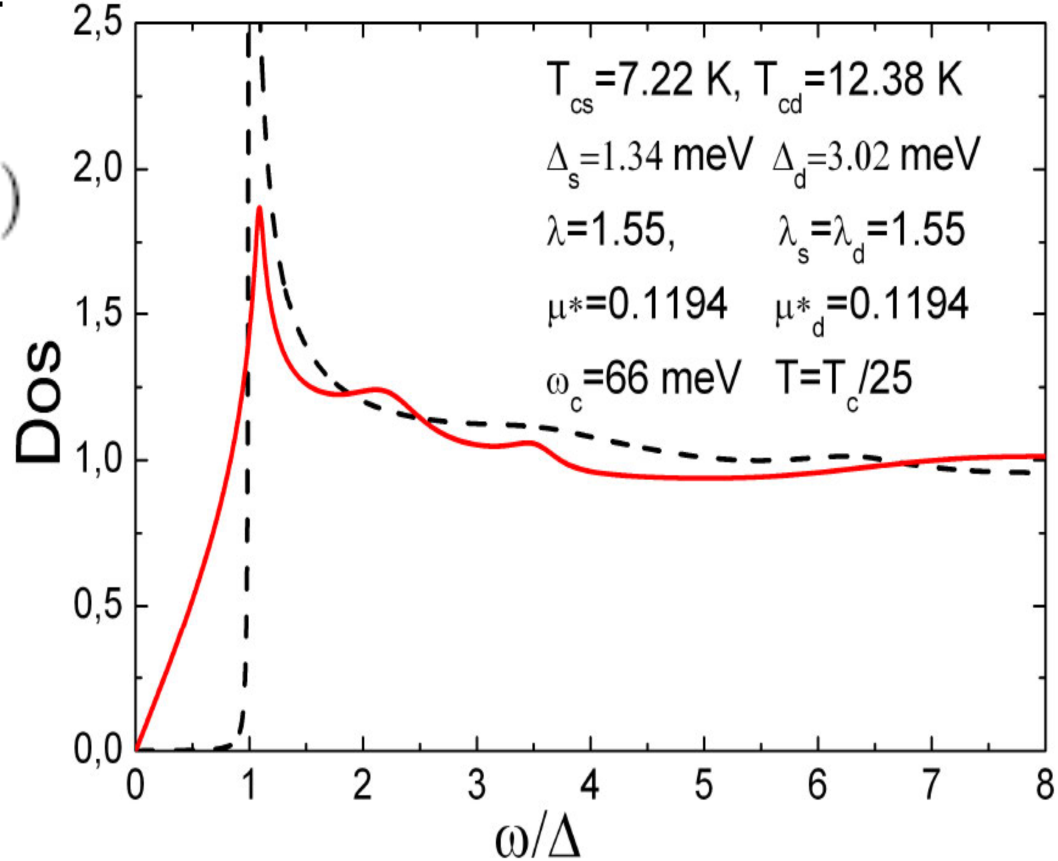
$$\alpha^2 F(\Omega, \phi, \phi') = \alpha^2 F_s(\Omega) + \alpha^2 F_d(\Omega)\sqrt{2}cos(2\phi)\sqrt{2}cos(2\phi')$$

$$\mu^*(\phi, \phi') = \mu^*_s + \mu^*_d(\Omega)\sqrt{2}cos(2\phi)\sqrt{2}cos(2\phi').$$

$$\Delta(\omega, \phi') = \Delta_d(\omega)cos(2\phi)$$

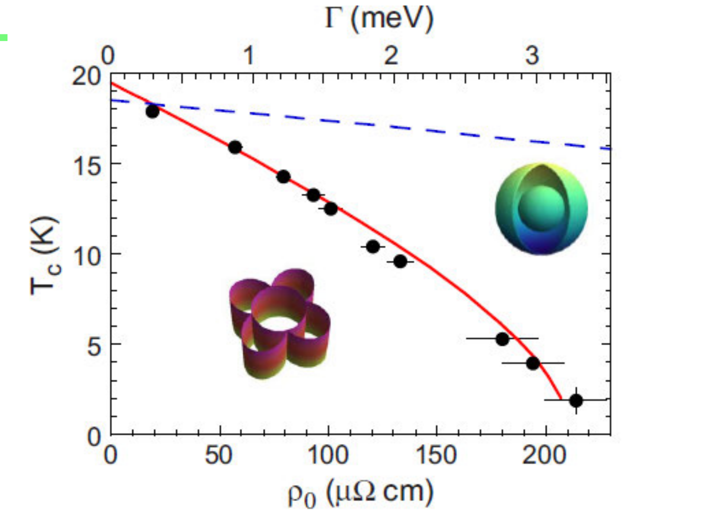
$$Z(\omega, \phi') = Z_s(\omega)$$

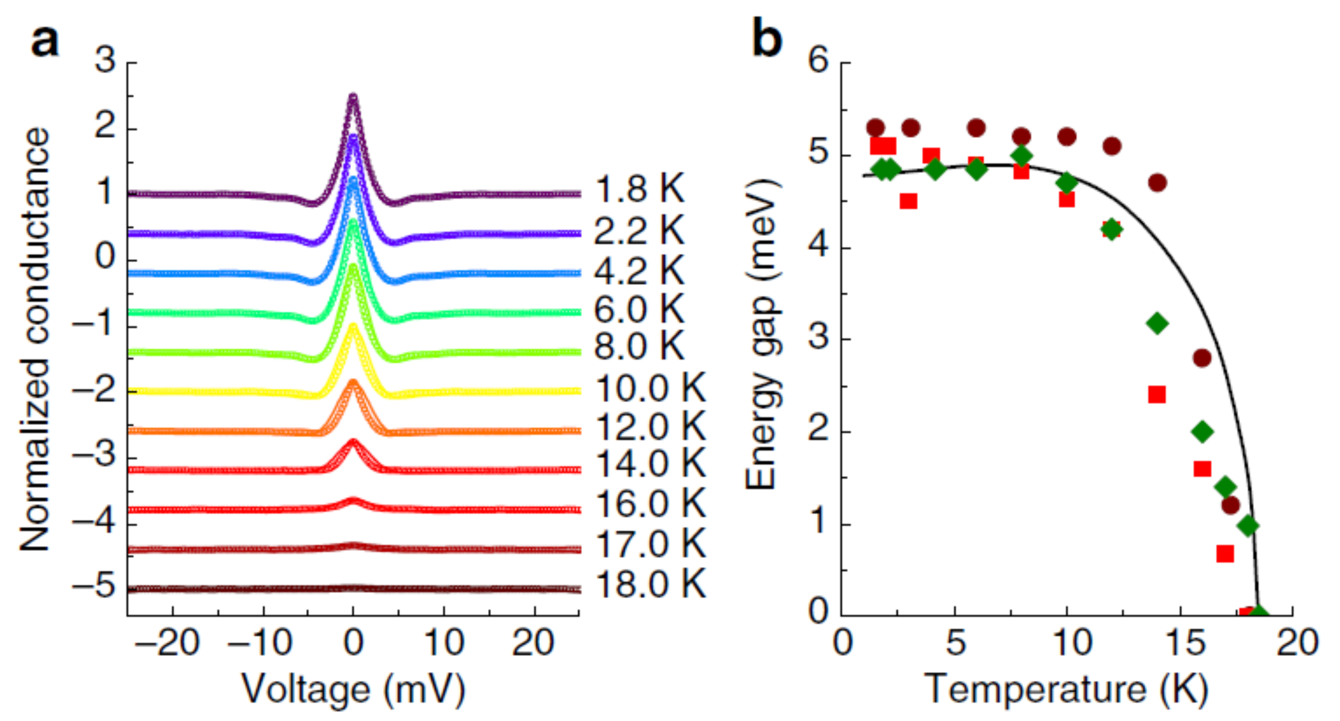
$$N_d(\omega) = \int_0^{2\pi} \frac{d\phi}{2\pi} \Re\left\{\frac{\omega}{\sqrt{\omega^2 - \Delta_d^2(\omega)cos^2(2\phi)}}\right\}$$



E. Schachinger and J.P. Carbotte, *Extended Eliashberg theory for d-wave superconductivity and application to cuprate*, Ed J.K. Sadasiva, S.M. Rao, *Models and Methods of High-Tc Superconductivity: Some Frontal Aspects*, Nova Science Publisher (2003) Vol. II.

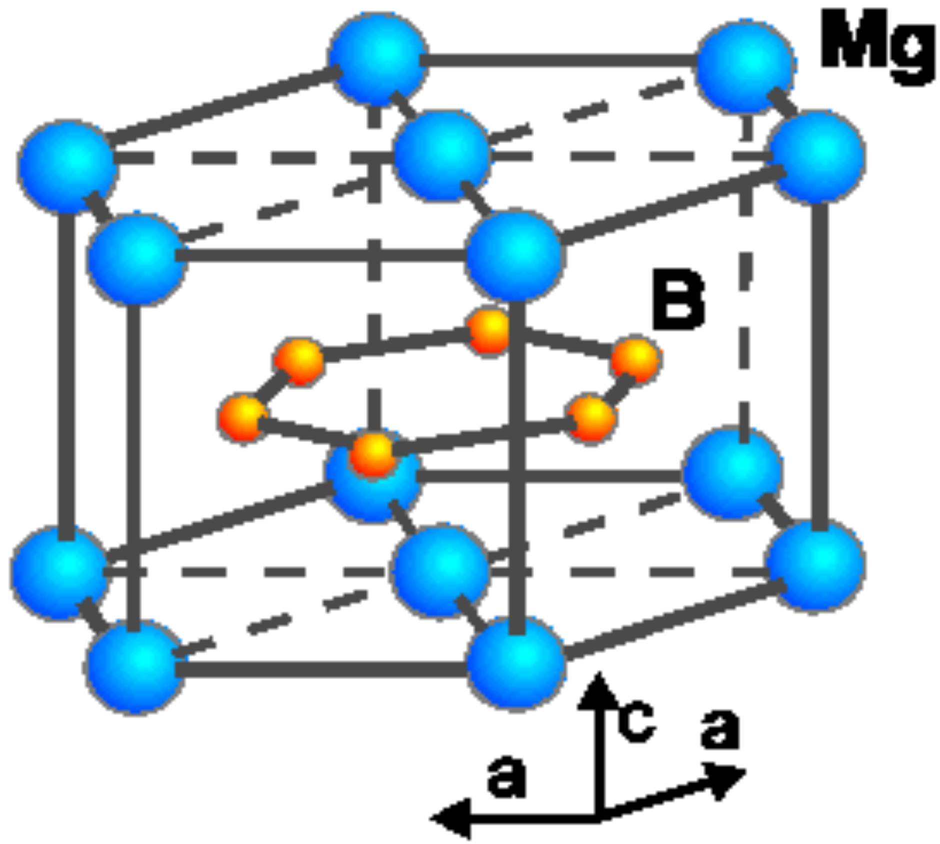
PuCoGa5 Tc=18 K





F. Jutier, G.A. Ummarino, J.C. Griveau, F. Wastin, E. Colineau, J. Rebizant, N. Magnani and R. Caciuffo, Phys. Rev. B 77, 024521 (2008); D. Daghero, M. Tortello, G.A. Ummarino, J.-C. Griveau, E. Colineau, A.B. Shick, R. Caciuffo, J. Kolorenc and A.I. Lichtenstein, Nature Communications 3, 786, (2012)

Magnesium diboride
MgB_2 T_c=39.4 K (2001)



Two conduction bands: π and σ

Four spectral functions

Four Coulomb pseudopotential

Eliashberg Equations

↓

Two isotropic order parameters

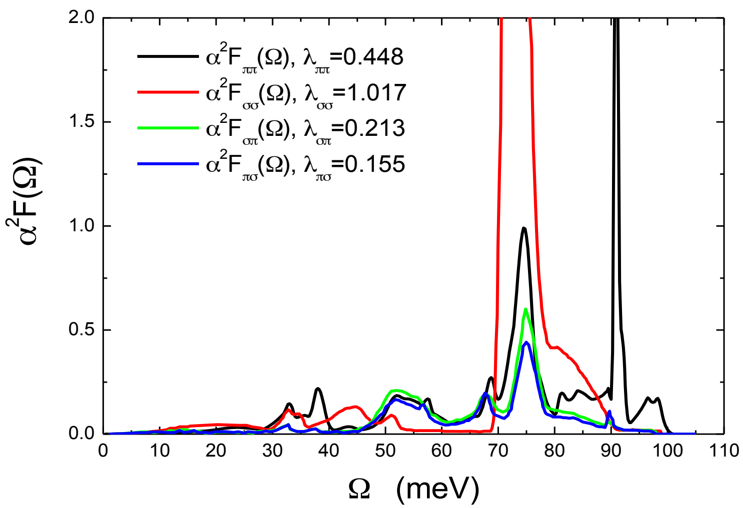
Two renormalization functions

$$
\begin{aligned}
\omega_n Z_i(i\omega_n) &= \omega_n + \pi T \sum_{m,j} \lambda_{ij}(i\omega_n, i\omega_m) N_j^Z(i\omega_m) + \\
&+ \sum_j \left[\Gamma_{\mathrm{ij}} + \Gamma_{\mathrm{ij}}^M\right] N_j^Z(i\omega_n) \\
Z_i(i\omega_n)\Delta_i(i\omega_n) &= \pi T \sum_{m,j} \left[\lambda_{ij}(i\omega_n, i\omega_m) - \mu_{ij}^*(\omega_c)\right] \times \\
&\times \Theta(\omega_c - |\omega_m|) N_j^\Delta(i\omega_m) + \sum_j [\Gamma_{\mathrm{ij}} + \Gamma_{\mathrm{ij}}^M] N_j^\Delta(i\omega_n)
\end{aligned}
$$

$$
\lambda_{ij}(i\omega_m - i\omega_n) \equiv 2 \int_0^\infty d\Omega \frac{\Omega\, \alpha_{ij}^2 F(\Omega)}{\Omega^2 + (\omega_n - \omega_m)^2}
$$

$$
N_j^\Delta(i\omega_m) = \Delta_j(i\omega_m) \cdot \left[\sqrt{\omega_m^2 + \Delta_j^2(i\omega_m)}\right]^{-1}, \qquad N_j^Z(i\omega_m) = \omega_m \cdot \left[\sqrt{\omega_m^2 + \Delta_j^2(i\omega_m)}\right]^{-1}
$$

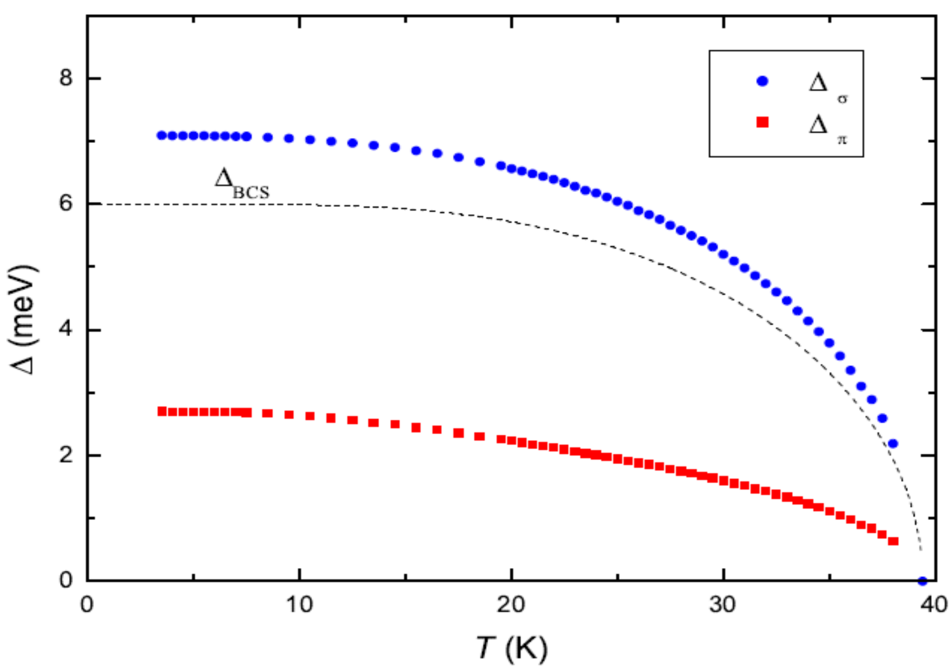
$$
\frac{\lambda_{ij}}{\lambda_{ji}} = \frac{N_i(0)}{N_j(0)}
$$



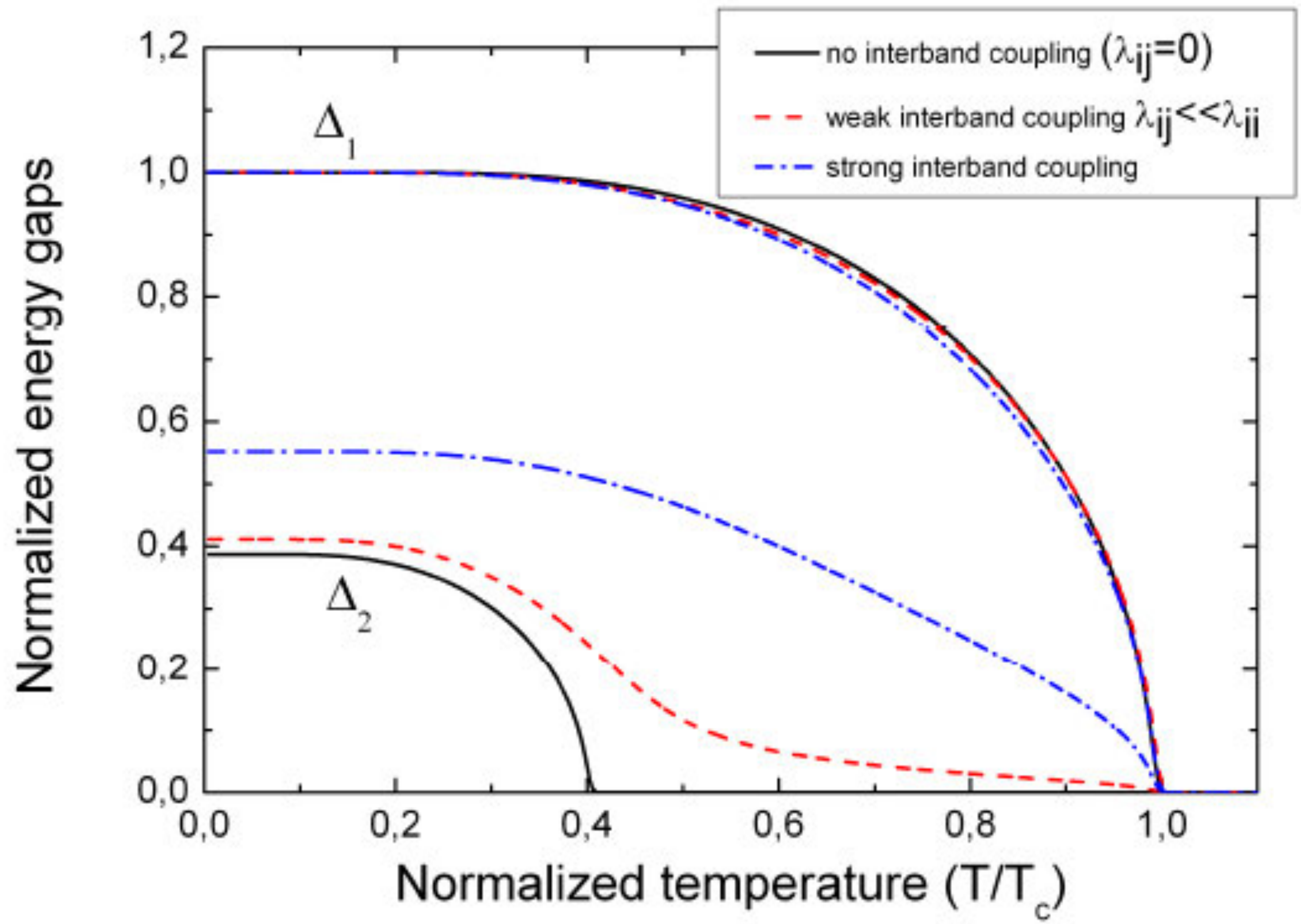
$$\mu^*_{\sigma\sigma} = 0.13,\ \mu^*_{\sigma\pi} = 0.042$$

$$\mu^*_{\pi\sigma} = 0.03,\ \mu^*_{\pi\pi} = 0.11$$

No free parameters!



All physical observables are in agreement with theoretical predictions!



E.J. Nicol and J.P. Carbotte, *Properties of the superconducting state in a two-band model* Phys. Rev. B **71**, 054501 (2005).

Proximity effect

$$\tilde{\Delta}_S(n) = \pi T_c \sum_m [\lambda_S^-(m-n) - \mu_S^*] \times \frac{\tilde{\Delta}_S(m)}{|\tilde{\omega}_S(m)|} + \Gamma_S \frac{\tilde{\Delta}_N(n)}{|\tilde{\omega}_N(n)|},$$

$$\tilde{\omega}_S(n) = \omega_n + \pi T_c \sum_m \lambda_S^+(m-n)\,\mathrm{sgn}\,\omega_m + \Gamma_S\,\mathrm{sgn}\,\omega_n$$

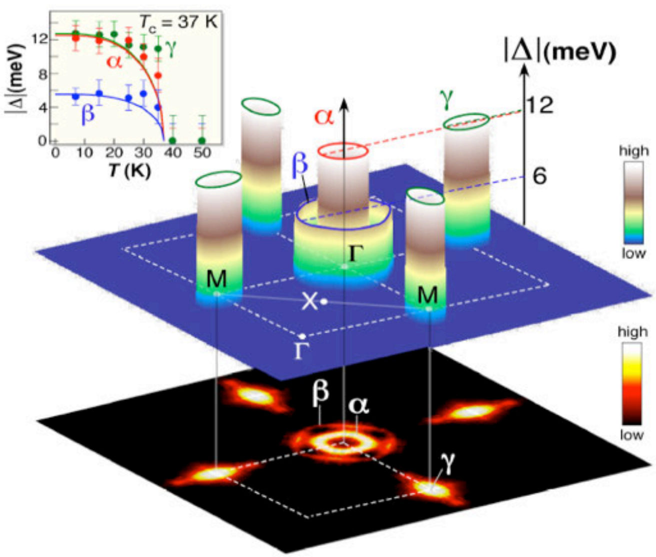
$$\tilde{\Delta}_N(n) = \pi T_c \sum_m [\lambda_N^-(m-n) - \mu_N^*] \times \frac{\tilde{\Delta}_N(m)}{|\tilde{\omega}_N(m)|} + \Gamma_N \frac{\tilde{\Delta}_S(n)}{|\tilde{\omega}_S(n)|},$$

$$\tilde{\omega}_N(n) = \omega_n + \pi T_c \sum_m \lambda_N^+(m-n)\,\mathrm{sgn}\,\omega_m + \Gamma_N\,\mathrm{sgn}\,\omega_n$$

H. G. Zarate and J. P. Carbotte, PRB 35, 3256 (1987)

Superconductivity in Fe-based compounds

The superconductivity in Fe-based layered compound is unconventional and mediated by antiferromagnetic spin fluctuations. This resulting state is an example of extended s-wave pairing with a sign reversal of the order parameter between different Fermi surface sheets.



$Ba_{0.6}K_{0.4}Fe_2As_2$: H. Ding et al, Europhys. Lett. 83 (2008)

3 bands with s-wave gap

The phases of two of the three gaps are opposite

Intraband coupling provided by phonons (P)

Interband coupling provided by AFM spin fluctuations (SF)

$$\lambda^{P}{}_{ii} >> \lambda^{P}{}_{ij} \approx 0$$

Phonons mainly provide intraband coupling

$$\lambda^{P}{}_{ii} < 0.35$$

$$\lambda^{SF}{}_{ii} << \lambda^{SF}{}_{ij}$$

The SF mainly provide the interband coupling

Three-band s± Eliashberg equations

$$\Delta_i(i\omega_n)Z_i(i\omega_n) =$$

$$\pi T \sum_j \sum_{m=-\infty}^{+\infty} \left[\Lambda^P{}_{ij}(i\omega_n - i\omega_m) - \Lambda^S{}_{ij}(i\omega_n - i\omega_m) - \mu^*{}_{ij}(\omega_c)\right] N_\Delta{}^j(i\omega_m)\theta(\omega_c - |\omega_m|) +$$

$$+ \pi T \sum_j (\Gamma_{ij}{}^N - \Gamma_{ij}{}^M) N_\Delta{}^j(i\omega_n)$$

Multi-band s-wave **imaginary- axis** Eliashberg equations with phonons (P) and spin fluctuations (S)

i,j =1-3 band index

$$\omega_n Z_i(i\omega_n) =$$

$$\omega_n + \pi T \sum_j \sum_{m=-\infty}^{+\infty} \left[\Lambda^P{}_{ij}(i\omega_n - i\omega_m) + \Lambda^S{}_{ij}(i\omega_n - i\omega_m)\right] N_Z{}^j(i\omega_m) +$$

$$+ \pi T \sum_j (\Gamma_{ij}{}^N + \Gamma_{ij}{}^M) N_Z{}^j(i\omega_n)$$

Three-band s± Eliashberg equations

$$N_Z{}^j(i\omega_m) = \frac{\omega_m Z_j(i\omega_m)}{\sqrt{\omega_m{}^2 Z_j{}^2(i\omega_m) + Z_j{}^2(i\omega_m)\Delta_j{}^2(i\omega_m)}},$$

$$N_\Delta{}^j(i\omega_m) = \frac{\Delta_j(i\omega_m)}{\sqrt{\omega_m{}^2 Z_j{}^2(i\omega_m) + Z_j{}^2(i\omega_m)\Delta_j{}^2(i\omega_m)}}$$

$$\Lambda^P{}_{ij}\left(i\omega_n - i\omega_m\right) = 2\int_0^{+\infty} d\Omega\, \frac{\alpha_{ij}{}^2(\Omega)F(\Omega)}{\Omega^2 + (i\omega_n - i\omega_m)^2}$$

$$\Lambda^S{}_{ij}\left(i\omega_n - i\omega_m\right) = 2\int_0^{+\infty} d\Omega\, \frac{P_{ij}(\Omega)}{\Omega^2 + (i\omega_n - i\omega_m)^2}$$

$$\lambda^P{}_{ij} = 2\int_0^{+\infty} d\Omega\, \frac{\alpha_{ij}{}^2(\Omega)F(\Omega)}{\Omega}$$

$$\lambda^S{}_{ij} = 2\int_0^{+\infty} d\Omega\, \frac{P_{ij}(\Omega)}{\Omega}$$

Inputs of the three-band Eliashberg equations

$\alpha^2 F_{ij}(\Omega)$	Electron-phonon spectral functions (9)
$P_{ij}(\Omega)$	Antiferromagnetic spin fluctuactions spectral functions (9)
$\mu^*{}_{ij}$	Coulomb pseudopotential matrix elements (9)
ω_c	electronic cut-off energy
$\Gamma^{N,M}{}_{ij}$	non-magnetic (N) and magnetic (M) impurity scattering rates (9+9)
$N_i(0)$	normal density of states at the Fermi level of the i-th band (3)

Assumptions of the three-band Eliashberg model

To reduce the number of free parameters we exploit the fact that

λ_{ii}^{ph} >> $\lambda_{ij}^{ph} \approx 0$ phonons mainly provide *intra*band coupling
λ_{ij}^{sf} >> $\lambda_{ii}^{sf} \approx 0$ spin fluctuations mainly provide *inter*band coupling

	$N_1(0)/N_2(0)$	$N_1(0)/N_3(0)$	$N_2(0)/N_3(0)$
BaFeCoAs	1.12	4.50	
LaFeAsOF		0.91	0.53
BaKFeAs		1.00	2.00
SmFeAsOF		2.50	2.00

Assumptions:

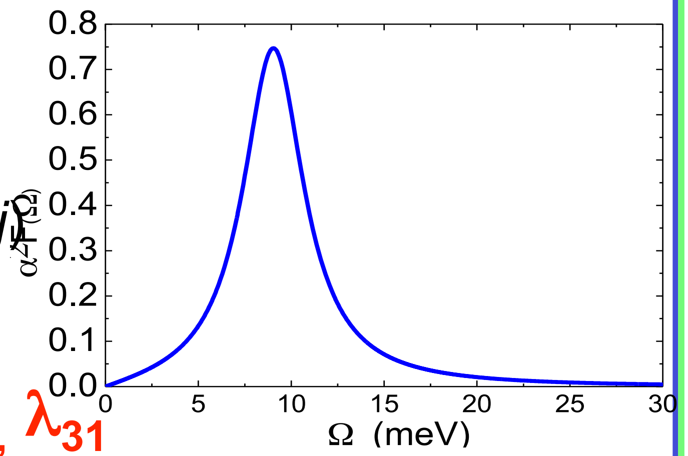
1) $\lambda_{ii}^{sf} = 0$ (no intraband SF coupling)
2) $\mu^*_{ij}=\mu^*_{ii} =0$ (no Coulomb pseudopotential)
3) $\Gamma_{ij}=0$ (no impurities)
4) $\lambda_{ii}^{ph} = \lambda_{ij}^{ph} = 0$ (no phonon coupling)

5) $\alpha_{ij}^2F(\Omega)=C_{ij}\{1/[(\Omega-\Omega_{ij})^2+Y^2_{ij}]-1/[(\Omega+\Omega_{ij})^2+Y^2_{ij}]\}$

6) $\Omega_{ij}=\Omega_0=2Tc/5$ and $Y_{ij}=\Omega_0/2$ (empirical law)

(The peak in the spectrum is the same for all *i,j*)

7) $N_i(0)$ from band theory calculations

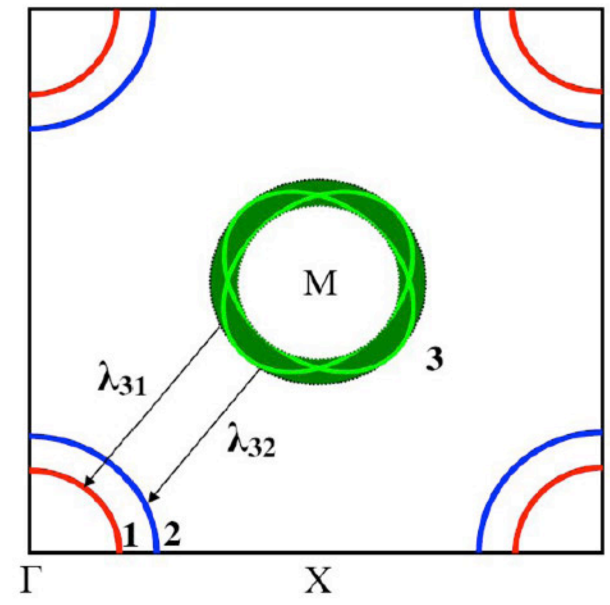


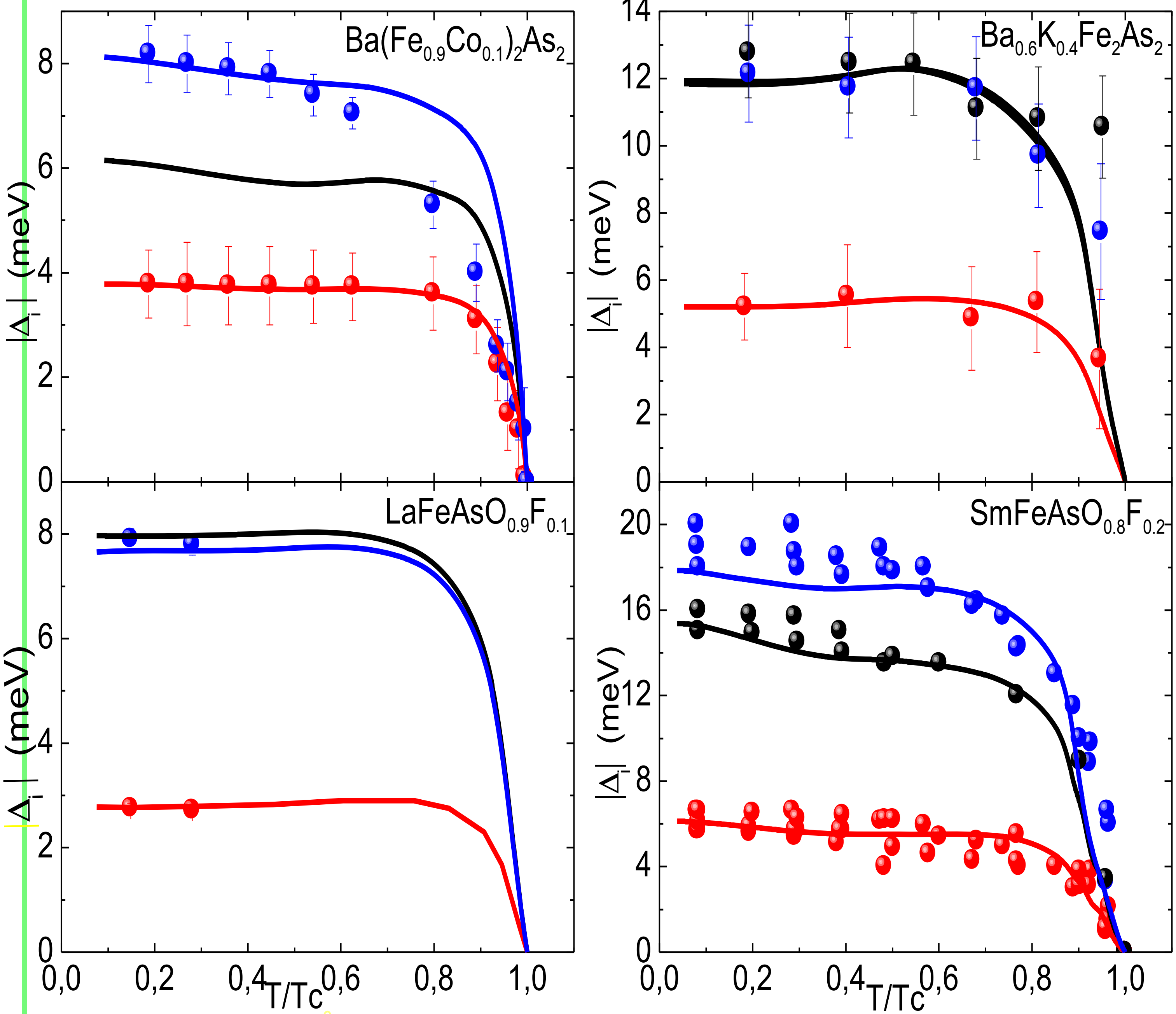
Only two free parameters: λ_{31}, λ_{32} or λ_{21}, λ_{31}

The coupling matrix λ_{ij} is thus:

$$\lambda_{ij}=\begin{pmatrix} 0 & \lambda_{21}\nu_{21} & \lambda_{31}\nu_{31} \\ \lambda_{21} & 0 & \lambda_{32}\nu_{32} \\ \lambda_{31} & \lambda_{32} & 0 \end{pmatrix}$$

where
$\nu_{31}=N_3(0)/N_1(0)$,
$\nu_{32}=N_3(0)/N_2(0)$
$\nu_{21}=N_2(0)/N_1(0)$





T_{cexp}(K)	T_{cth}(K)
22.6	23.7 λ=2.8
28.6	26.5 λ=2.4
37.0	38.3 λ=2.8
52.0	52.8 λ=2.4

G.A. Ummarino, Phys. Rev. B 83, 092508 (2011)

Imaginary-axis Eliashberg equations for the upper critical field

$$\omega_n Z_i(i\omega_n) = \omega_n + \pi T \sum_{m,j} \Lambda_{ij}(i\omega_n - i\omega_m)\mathrm{sign}(\omega_m)$$

$$Z_i(i\omega_n)\Delta_i(i\omega_n) = \pi T \sum_{m,j} [\Lambda_{ij}(i\omega_n - i\omega_m) - \mu^*_{ij}(\omega_c)] \cdot$$
$$\cdot \theta(|\omega_c| - \omega_m)\chi_j(i\omega_m) Z_j(i\omega_m)\Delta_j(i\omega_m)$$

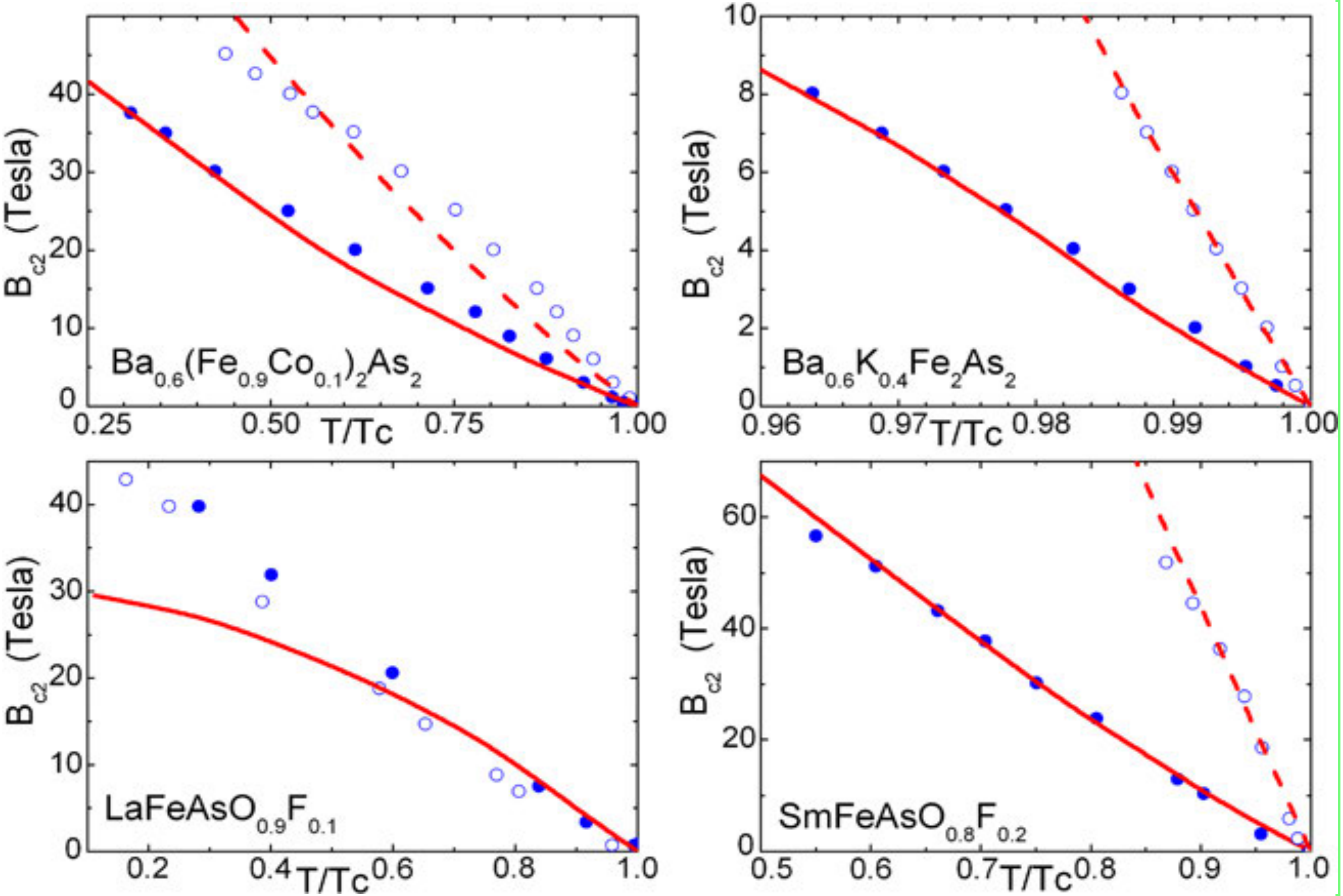
$$\chi_j(i\omega_m) = (2/\sqrt{\beta_j}) \int_0^{+\infty} dq \exp(-q^2) \cdot \qquad \beta_j = \pi H_{c2} v_{Fj}^2/(2\Phi_0)$$
$$\cdot \tan^{-1}\left[\frac{q\sqrt{\beta_j}}{|\omega_m Z_j(i\omega_m)| + i\mu_B H_{c2}\mathrm{sign}(\omega_m)}\right]$$

Three free parameters: the Fermi velocities that can be calculated by DFT but if they haven't been yet calculated

Free electron model: $v_{Fj} \propto N_{\mathrm{N}}^{j}(0)$

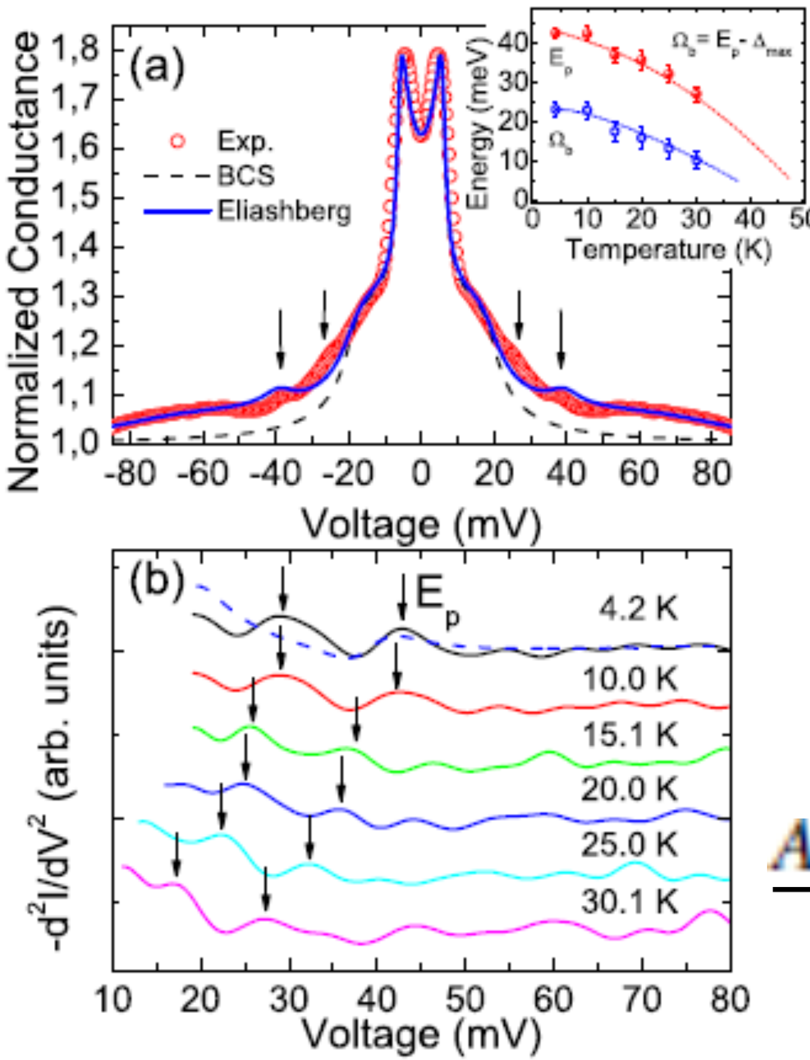
so that $v_{F2} = v_{F1}\nu_2/\nu_1$ and $v_{F3} = v_{F1}/\nu_1$

Only a free parameter: v_{F1}



Ba0.6(Fe0.9Co0.1)2As2
Ba0.6K0.4Fe2As2
LaFeAsO0.9F0.1
SmFeAsO0.8F0.2
Bc2 (Tesla)
T/Tc

Andreev reflection: mechanism of superconductivity in the iron pnictides



Stuctures present at

$$\Omega_0 + \Delta_{max}$$

We put the solution of Eliashberg equations

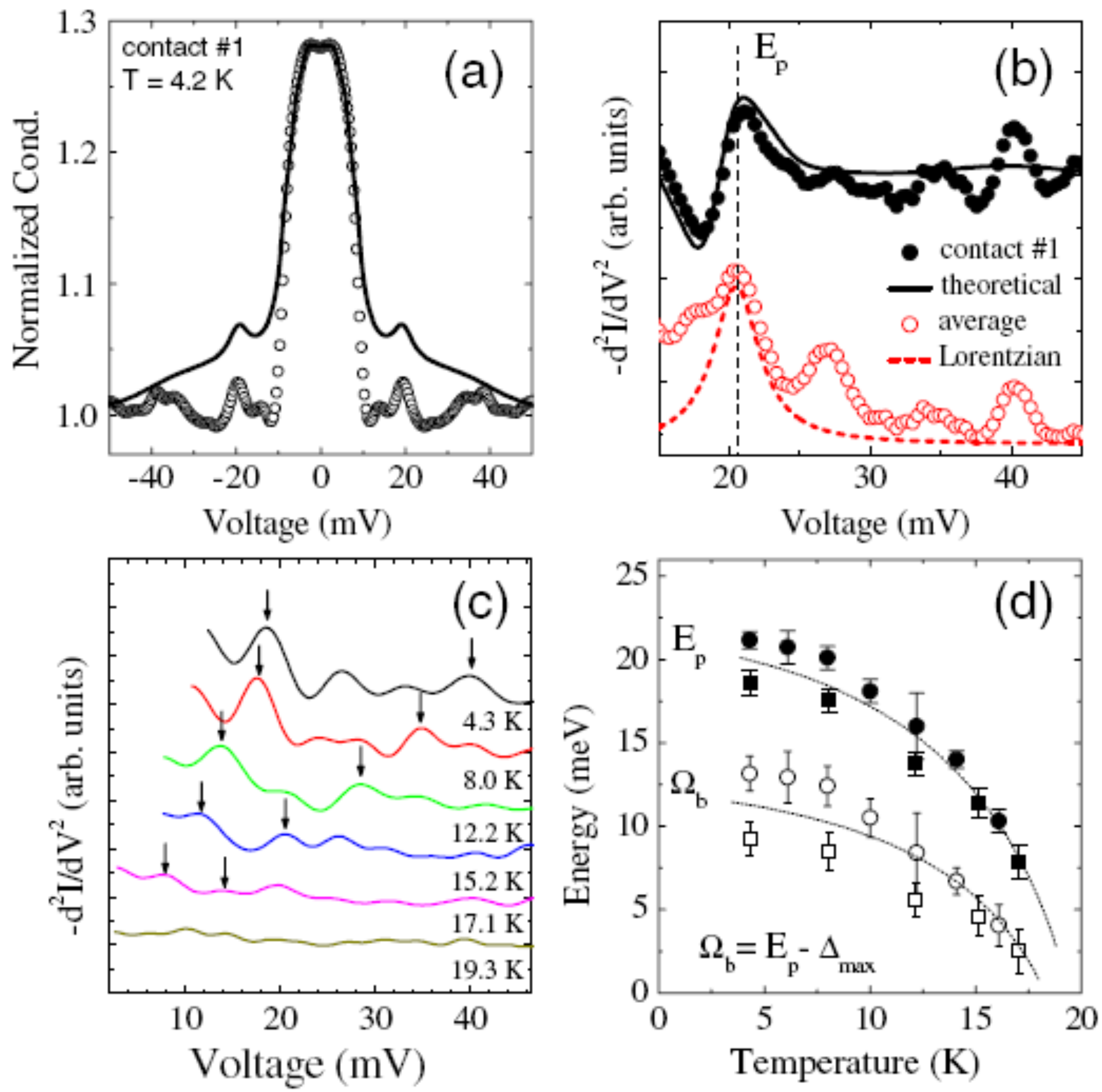
in the Andreev reflection equation of conductance:

bosonic structures appear!

$Ag/SmFeAsO_{0.8}F_{0.2}$ *point-contact.*

D. Daghero, M. Tortello, G.A. Ummarino and R.S. Gonnelli, Rep. Prog. Phys. 74, 124509 (2011)

Point-Contact Andreev-Reflection Study of $Ba(Fe_{1-x}Co_x)_2As_2$ Single Crystals

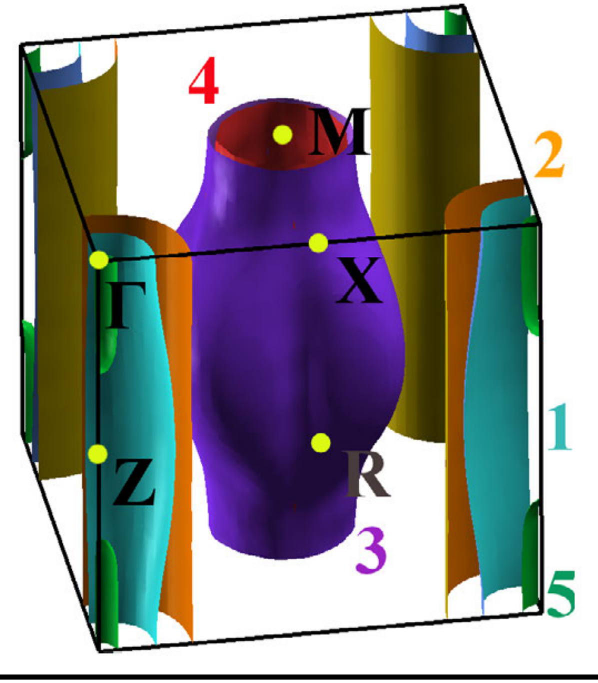


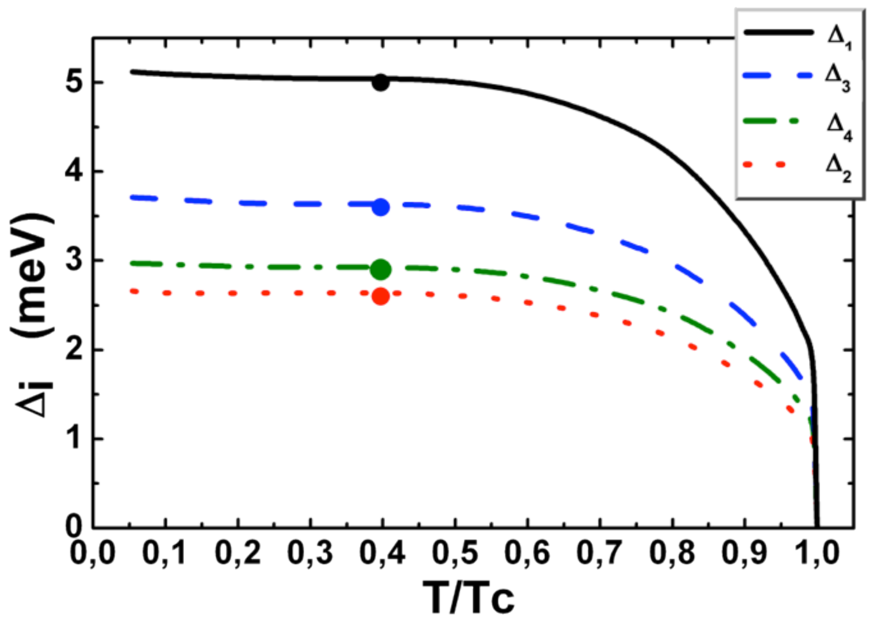
M. Tortello, D. Daghero, G. A. Ummarino, V. A. Stepanov, J. Jiang, J. D. Weiss,

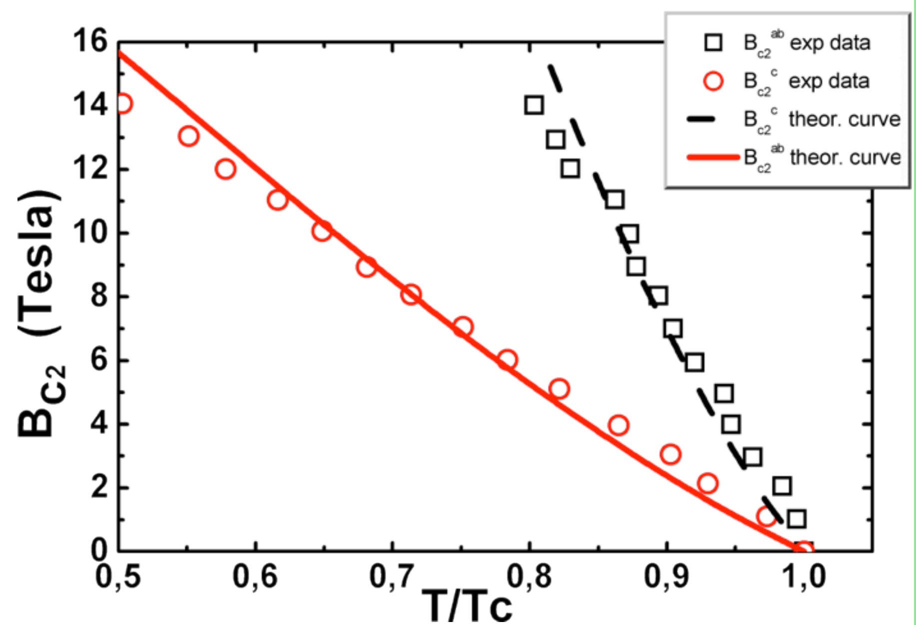
E. E. Hellstrom, and R. S. Gonnelli1, PRL 105, 237002 (2010)

LiFeAs Tc=18 K

$$\lambda_{ij} = \begin{pmatrix} \lambda_{11} & 0 & \lambda_{13} & \lambda_{14} \\ 0 & 0 & \lambda_{23} & \lambda_{24} \\ \lambda_{31} = \lambda_{13}\nu_{13} & \lambda_{32} = \lambda_{23}\nu_{23} & 0 & 0 \\ \lambda_{41} = \lambda_{14}\nu_{14} & \lambda_{42} = \lambda_{24}\nu_{24} & 0 & 0 \end{pmatrix}$$

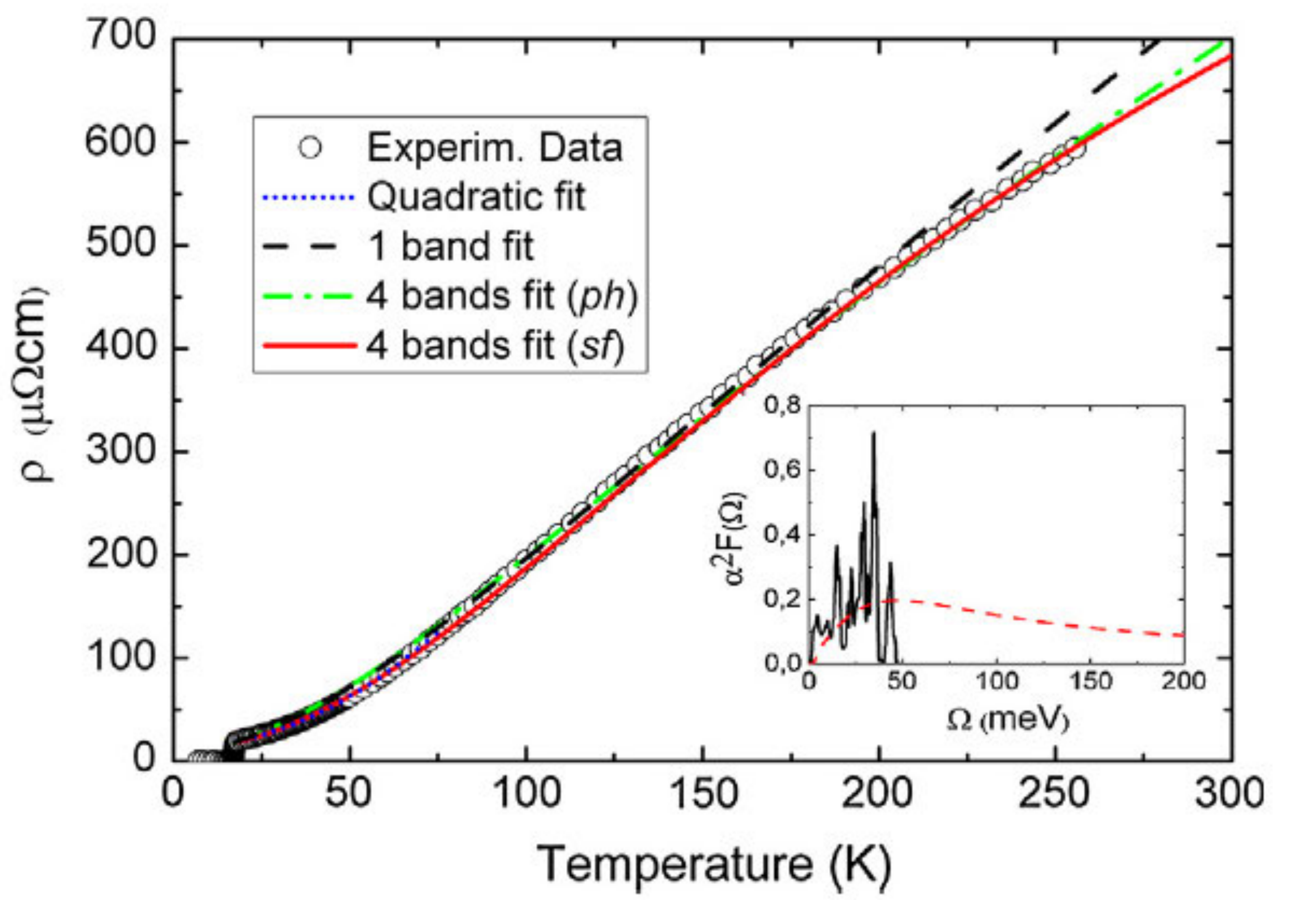






G A Ummarino, Sara Galasso and A Sanna J. Phys.: Condens. Matter 25 (2013) 205701

$\lambda_{tot,tr}$=0.77, $\lambda_{tot,sup}$=2 as HTCS



The same happens for BaFeCoAs, BaFeKAs…

Eliashberg theory works for:

All old superconductors

fullerenes

BKBO

MgB_2

HTCS overdoped

iron pnictides

$PuCoGa_5$ and,may be, for heavy

fermions and HTCS optimally doped

It does’ t work (now, in this form)

for HTCS underdoped

ESSENTIAL (not exaustive!) BIBLIOGRAPHY OF ELIASHBERG THEORY

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4) F. Marsiglio, J.P. Carbotte, in: K.H. Bennemann, J.B. Ketterson (Ed.), The Physics of Conventional and Unconventional Superconductors, Springer-Verlag;

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7) M.L. Kulic, Physics Reports 338, 1 (2000); E.G. Maksimov, M.L. Kulic, and O.V. Dolgov, Advances in Condensed Matter Physics 2010, 423725 (2010)