Exercise Sheet 9

1. *Clebsch-Gordan coefficients*

Prove the following recursion relation for the Clebsch-Gordan coefficients:

$$\begin{aligned}
&\sqrt{(j\pm m+1)(j\mp m)}\,\langle j_a, m_a;\, j_b, m_b | j, m\pm 1;\, j_a; j_b\rangle \\
&= \sqrt{(j_a\mp m_a+1)(j_a\pm m_a)}\,\langle j_a, m_a\mp 1;\, j_b, m_b | j, m;\, j_a; j_b\rangle \\
&\ +\sqrt{(j_b\mp m_b+1)(j_b\pm m_b)}\,\langle j_a, m_a;\, j_b, m_b\mp 1 | j, m;\, j_a; j_b\rangle
\end{aligned}$$

2. *Clebsch-Gordan coefficients*

Write a program that takes two angular momentum quantum numbers j_a and j_b as input and produces a matrix for transforming from the product states $|j_a, m_a;\, j_b, m_b\rangle$ to the total angular momentum states $|j, m\rangle$. Example:

$j_a = 1$	$j_b = 1/2$	$j = 3/2$				$j = 1/2$	
m_a	m_b	$m = 3/2$	$m = 1/2$	$m = -1/2$	$m = -3/2$	$m = 1/2$	$m = -1/2$
1	1/2	1					
1	$-1/2$		$\sqrt{1/3}$			$\sqrt{2/3}$	
0	1/2		$\sqrt{2/3}$			$-\sqrt{1/3}$	
0	$-1/2$			$\sqrt{2/3}$			$\sqrt{1/3}$
-1	1/2			$\sqrt{1/3}$			$-\sqrt{2/3}$
-1	$-1/2$				1		

Hint: Expand the square of the coefficients into a continued fraction.

3. *Addition of three angular momenta*

Consider three independent spin 1/2 systems with spin operators $\vec{S}_a$, $\vec{S}_b$, and $\vec{S}_c$. Add the spins in two different ways:

i. $(\vec{S}_a + \vec{S}_b) + \vec{S}_c$

ii. $\vec{S}_a + (\vec{S}_b + \vec{S}_c)$

Compare the results.