Exercise Sheet 6 due 5 December

1. *harmonic oscillator*

Rewrite the relations for the ladder operators from last week's exercise in the Dirac formalism as $\langle a\varphi|\psi\rangle = \langle\varphi|a^\dagger\psi\rangle$ and $\langle a^\dagger\varphi|\psi\rangle = \langle\varphi|a\psi\rangle$. Use this to show that for the norm (defined by $\|\psi\|^2 = \langle\psi|\psi\rangle$) we have $\|a\varphi\|^2 = \langle\varphi|a^\dagger a\varphi\rangle$ and $\|a^\dagger\varphi\|^2 = \langle\varphi|aa^\dagger\varphi\rangle$.
Given an eigenstate $|\varphi_n\rangle$ with $a^\dagger a|\varphi_n\rangle = n|\varphi_n\rangle$ show that

i. $a^\dagger|\varphi_n\rangle$ is an eigenvector of $a^\dagger a$ with eigenvalue $n{+}1$ and norm $\sqrt{n{+}1}$,

ii. $a|\varphi_n\rangle$ is an eigenvector of $a^\dagger a$ with eigenvalue $n{-}1$ and norm $\sqrt{n}$.

2. *expectation values*

Consider the normalized eigenstates $|n\rangle$ of a harmonic oscillator with $H|n\rangle = \hbar\omega(n{+}1/2)|n\rangle$ and $\langle n|m\rangle = \delta_{n,m}$. Calculate the expectation values of the momentum ($\langle n|p|n\rangle$) and its square ($\langle n|p^2|n\rangle$), where $p = -i\hbar\frac{d}{dx}$.

3. *momentum and translations*

i. Show that the momentum operator $\hat{p} = -i\hbar\frac{d}{dx}$ is Hermitian

ii. Calculate $e^{i\hat{p}\Delta x/\hbar}\,\varphi(x)$ for an arbitrary wave function $\varphi(x)$ and displacement Δx by expanding the exponential in a power series and resumming the resulting power series.