Exercise Sheet 5 due 28 November

1. *ladder operators*

Using the ladder operator $a^\dagger = (\xi - d/d\xi)/\sqrt{2}$ we can write the eigenstates of the Hamiltonian

$$H = -\frac{1}{2}\frac{d^2}{d\xi^2} + \frac{\xi^2}{2}$$

with eigenenergy $\varepsilon_n = n+1/2$ as

$$\varphi_n(\xi) = \frac{(a^\dagger)^n}{\sqrt{n!}}\,\varphi_0(\xi) \tag{1}$$

where the ground state is given by $\varphi_0(\xi) = e^{-\xi^2/2}/\sqrt[4]{\pi}$.

i. Show using integration by parts that the operators $a = (\xi + d/d\xi)/\sqrt{2}$ and $a^\dagger$ are related as

$$\int\limits_{-\infty}^{\infty} d\xi\,\overline{(a\varphi(\xi))}\,\psi(\xi) = \int\limits_{-\infty}^{\infty} d\xi\,\overline{\varphi(\xi)}\,(a^\dagger\psi(\xi)) \text{ and } \int\limits_{-\infty}^{\infty} d\xi\,\overline{(a^\dagger\varphi(\xi))}\,\psi(\xi) = \int\limits_{-\infty}^{\infty} d\xi\,\overline{\varphi(\xi)}\,(a\psi(\xi))$$

for arbitrary wave-functions $\varphi(\xi)$ and $\psi(\xi)$.

ii. Use this to show that for $H = a^\dagger a + 1/2$ we have

$$\int_{-\infty}^{\infty} d\xi\,\overline{(H\varphi(\xi))}\,\psi(\xi) = \int_{-\infty}^{\infty} d\xi\,\overline{\varphi(\xi)}\,(H\psi(\xi))$$

iii. Show that the eigenstates $\varphi_n(\xi)$ are orthonormal, i.e., that

$$\int_{-\infty}^{\infty} d\xi\,\overline{\varphi_n(\xi)}\varphi_m(\xi) = \delta_{n,m}$$

2. *Hermite polynomials*

Use (1) and $a^\dagger + a = \sqrt{2}\xi$ to verify the recurrence relation for the normalized eigenfunctions

$$\sqrt{2}\xi\varphi_n(\xi) = (a^\dagger + a)\varphi_n(\xi) = \sqrt{n+1}\varphi_{n+1}(\xi) + \sqrt{n}\,\varphi_{n-1}(\xi) \tag{2}$$

Given $\varphi_0(\xi) = e^{-\xi^2/2}/\sqrt[4]{\pi}$ find $\varphi_1(\xi)$ and $\varphi_2(\xi)$.

Most textbooks write the normalized eigenfunctions using the Hermite polynomials $H_n(\xi)$ as

$$\varphi_n(\xi) = \frac{H_n(\xi)}{\sqrt{2^n n!}}\frac{e^{-\xi^2/2}}{\sqrt[4]{\pi}}$$

Insert this expression into (2) to find the recurrence relation for the Hermite polynomials

$$H_{n+1}(\xi) = 2\xi H_n(\xi) - 2nH_{n-1}(\xi)$$