Functional renormalization group approach to interacting Fermi systems – DMFT as a booster rocket

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1. Introduction
2. Functional RG for Fermi systems
3. Two-dimensional Hubbard model
4. Leap to strong coupling: DMFT as a booster rocket

Review on fRG:

W. Metzner, M. Salmhofer, C. Honerkamp, V. Meden, and K. Schönhammer, Rev. Mod. Phys. **84**, 299 (2012)

1. Introduction

Interaction between (valence) electrons in solids $\Rightarrow$

- Spontaneous symmetry breaking (magnetic order, superconductivity)
- Correlation gaps without symmetry-breaking
  (e.g. Mott metal-insulator transition)
- Kondo effect
- Exotic liquids (*Luttinger liquids*, quantum critical systems)
- ...

The most striking phenomena involve electronic correlations beyond conventional mean-field theories (Hartree-Fock, LDA etc.).

Scale problem:

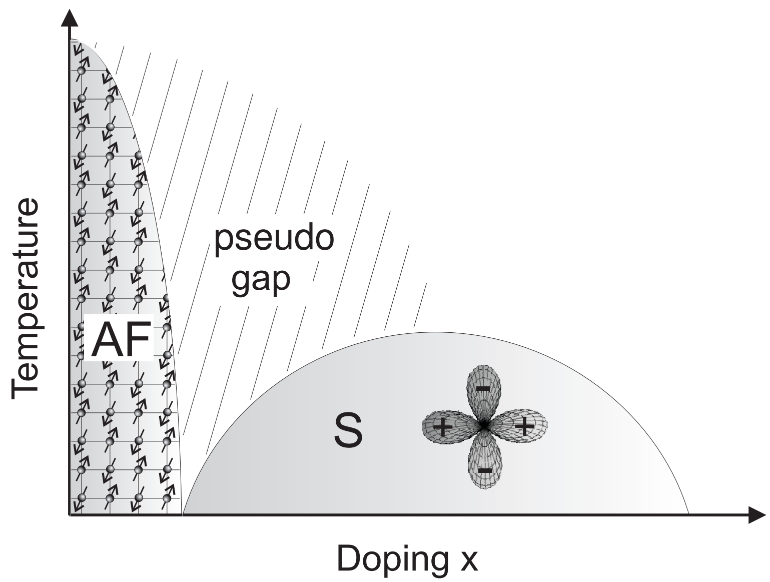
Very different behavior on different energy scales

Collective phenomena, coherence, and composite objects often emerge at scales far below bare energy scales of microscopic Hamiltonian

$\Longrightarrow$ PROBLEM

- for straightforward numerical treatments of microscopic systems
- for conventional many-body methods which treat all scales at once and within the same approximation (e.g. summing subsets of Feynman diagrams)

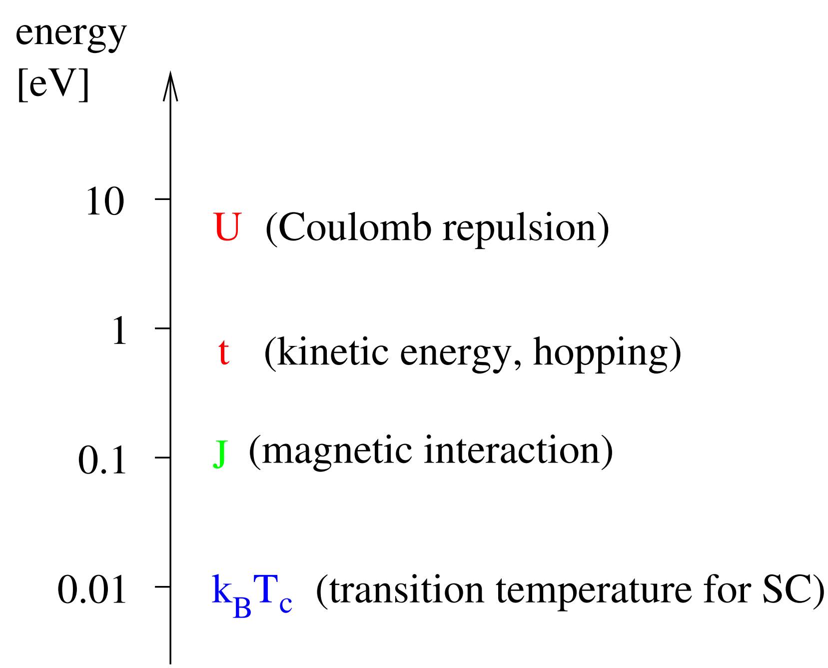
Example: High temperature superconductors



- antiferromagnetism
  in undoped compounds
- d-wave superconductivity
  at sufficient doping
- Pseudo gap, non-Fermi liquid
  in "normal" phase at finite T

Vast hierarchy of energy scales:

Magnetic interaction and superconductivity generated from kinetic energy and Coulomb interaction



Renormalization group idea

Strategy to deal with hierarchy of energy scales?

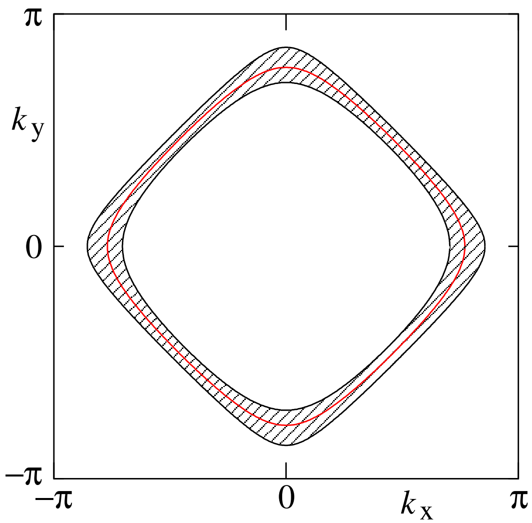
Main idea (Wilson):
Treat degrees of freedom with different energy scales successively, descending step by step from the highest scale.

In practice, using functional integral representation:
Integrate degrees of freedom (bosonic or fermionic fields) successively, following a suitable hierarchy of energy scales.

$\Rightarrow$ One-parameter family of effective actions Γ^Λ, interpolating smoothly between bare action and final effective action (for $\Lambda \to 0$) from which all physical properties can be extracted.

Advantage:
Small steps from scale Λ to $\Lambda' < \Lambda$ easier to control than going from highest scale Λ_0 to $\Lambda = 0$ in one shot.

Effective actions Γ^{Λ} can be defined for example by integrating only fields with momenta satisfying $|\epsilon_{\mathbf{k}} - \mu| > \Lambda$, which excludes a momentum shell around the Fermi surface.



Momentum space region around the Fermi surface excluded by a sharp momentum cutoff in a 2D lattice model

History of RG for Fermi systems:

Long tradition in 1D systems, starting in 1970s (Solyom, ...);
mostly field-theoretical RG with few couplings.

RG work for 2D or 3D Fermi systems with renormalization of interaction functions started in 1990s and can be classified as

- rigorous:
  Feldman, Trubowitz, Knörrer, Magnen, Rivasseau, Salmhofer;
  Benfatto, Gallavotti; ...

- pedagogical:
  Shankar; Polchinski; ...

- computational (using “functional RG”):
  Zanchi, Schulz; Halboth, Metzner; Honerkamp, Salmhofer, Rice; ...

2. Functional RG for Fermi systems

A natural way of dealing with many energy scales in interacting electron systems and a powerful source of new approximations.

- applicable to microscopic models (not only field theory)
- RG treatment of infrared singularities built in
- consistent fusion of distinct scale-dependent approximations

2.1. Effective action

Textbook: Negele & Orland, *Quantum Many-Particle Systems*

Interacting Fermi system with bare action

$$\mathcal{S}[\psi,\bar\psi] = -(\bar\psi, G_0^{-1}\psi) + V[\psi,\bar\psi]$$

$\psi_K, \bar\psi_K$ Grassmann variables, $K =$ quantum numbers + Matsubara frequency

Spin-$\frac{1}{2}$ fermions with momentum $\mathbf{k}$ and spin orientation σ: $K = (k_0, \mathbf{k}, \sigma)$

Bare propagator in case of translation and spin-rotation invariance:

$$G_0(K;K') = \delta_{KK'}\, G_0(k_0,\mathbf{k}), \text{ where } G_0(k_0,\mathbf{k}) = \frac{1}{ik_0 - (\epsilon_{\mathbf{k}} - \mu)}$$

Two-particle interaction:

$$V[\psi,\bar\psi] = \frac{1}{4}\sum_{K_1,K_2}\sum_{K_1',K_2'} V(K_1',K_2';K_1,K_2)\,\bar\psi_{K_1'}\psi_{K_1}\bar\psi_{K_2'}\psi_{K_2}$$

Generating functional for connected Green functions

$$\mathcal{G}[\eta,\bar\eta] = -\log\left\{\int\prod_K d\psi_K d\bar\psi_K\, e^{-\mathcal{S}[\psi,\bar\psi]}\, e^{(\bar\eta,\psi)+(\bar\psi,\eta)}\right\}$$

Connected m-particle Green function

$$\begin{aligned} G^{(2m)}(K_1,\ldots,K_m;K'_1,\ldots,K'_m) &= \\ -\underbrace{\langle\psi_{K_1}\ldots\psi_{K_m}\bar\psi_{K'_m}\ldots\bar\psi_{K'_1}\rangle_c}_{\text{connected average}} &= \frac{\partial^m}{\partial\eta_{K'_1}\ldots\partial\eta_{K'_m}}\frac{\partial^m}{\partial\bar\eta_{K_m}\ldots\partial\bar\eta_{K_1}}\mathcal{G}[\eta,\bar\eta]\bigg|_{\eta=\bar\eta=0}\end{aligned}$$

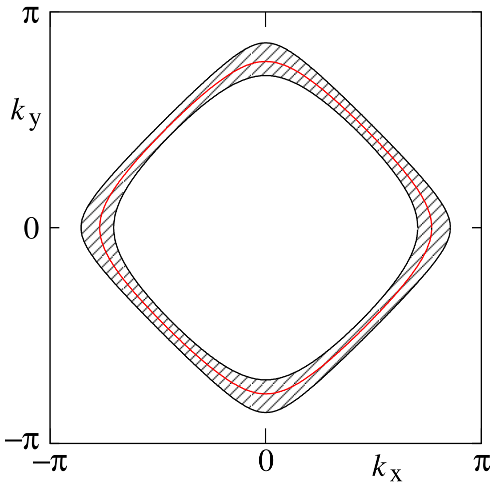
Legendre transform of $\mathcal{G}[\eta,\bar\eta]$: effective action

$$\Gamma[\psi,\bar\psi] = \mathcal{G}[\eta,\bar\eta] + (\bar\psi,\eta) + (\bar\eta,\psi) \quad \text{with} \quad \psi = -\frac{\partial\mathcal{G}}{\partial\bar\eta} \quad \text{and} \quad \bar\psi = \frac{\partial\mathcal{G}}{\partial\eta}$$

generates one-particle irreducible vertex functions $\Gamma^{(2m)}$

2.2. Exact flow equations

Impose infrared cutoff at energy scale $\Lambda > 0$, e.g. a momentum cutoff



$$G_0^\Lambda(k_0, \mathbf{k}) = \frac{\Theta^\Lambda(\mathbf{k})}{ik_0 - \xi_\mathbf{k}} \quad , \qquad \xi_\mathbf{k} = \epsilon_\mathbf{k} - \mu$$

$$\Theta^\Lambda(\mathbf{k}) = \Theta(|\xi_\mathbf{k}| - \Lambda)$$

Cutoff regularizes divergence of $G_0(k_0, \mathbf{k})$ in $k_0 = 0$, $\xi_\mathbf{k} = 0$ (Fermi surface)

Other choices: smooth cutoff, frequency cutoff

Cutoff excludes integration variables below scale Λ from functional integral

$\Rightarrow$ Λ-dependent functionals $\mathcal{G}^\Lambda[\eta, \bar\eta]$ and $\Gamma^\Lambda[\psi, \bar\psi]$.

Functionals $\mathcal{G}$ and Γ recovered for $\Lambda \to 0$.

Exact flow equation for Γ^Λ:

$$\boxed{\frac{d}{d\Lambda}\Gamma^\Lambda[\psi,\bar\psi] = -(\bar\psi, \dot{Q}_0^\Lambda \psi) - \frac{1}{2}\mathrm{tr}\left[\dot{\mathbf{Q}}_0^\Lambda \left(\mathbf{\Gamma}^{(2)\Lambda}[\psi,\bar\psi]\right)^{-1}\right]}$$

$$Q_0^\Lambda = (G_0^\Lambda)^{-1} \qquad \dot{Q}_0^\Lambda = \partial_\Lambda Q_0^\Lambda$$

$$\mathbf{Q}_0^\Lambda = \begin{pmatrix} Q^\Lambda_{0,KK'} & 0 \\ 0 & -Q^\Lambda_{0,K'K} \end{pmatrix} \qquad \mathbf{\Gamma}^{(2)\Lambda}[\psi,\bar\psi] = \begin{pmatrix} \dfrac{\partial^2\Gamma^\Lambda}{\partial\bar\psi_K\partial\psi_{K'}} & \dfrac{\partial^2\Gamma^\Lambda}{\partial\bar\psi_K\partial\bar\psi_{K'}} \\ \dfrac{\partial^2\Gamma^\Lambda}{\partial\psi_K\partial\psi_{K'}} & \dfrac{\partial^2\Gamma^\Lambda}{\partial\psi_K\partial\bar\psi_{K'}} \end{pmatrix}$$

Wetterich 1993; Salmhofer & Honerkamp 2001

Derivation: simple, see lecture notes!

Expansion in fields:

$$\mathbf{\Gamma}^{(2)\Lambda}[\psi,\bar\psi] = (\mathbf{G}^\Lambda)^{-1} - \tilde{\mathbf{\Sigma}}^\Lambda[\psi,\bar\psi]$$

where $\quad \mathbf{G}^\Lambda = \left(\mathbf{\Gamma}^{(2)\Lambda}[\psi,\bar\psi]\big|_{\psi=\bar\psi=0}\right)^{-1} = \begin{pmatrix} G^\Lambda_{KK'} & 0 \\ 0 & -G^\Lambda_{K'K} \end{pmatrix}$

$\tilde{\mathbf{\Sigma}}^\Lambda[\psi,\bar\psi]$ contains all contributions to $\mathbf{\Gamma}^{(2)\Lambda}[\psi,\bar\psi]$ which are at least quadratic in the fields.

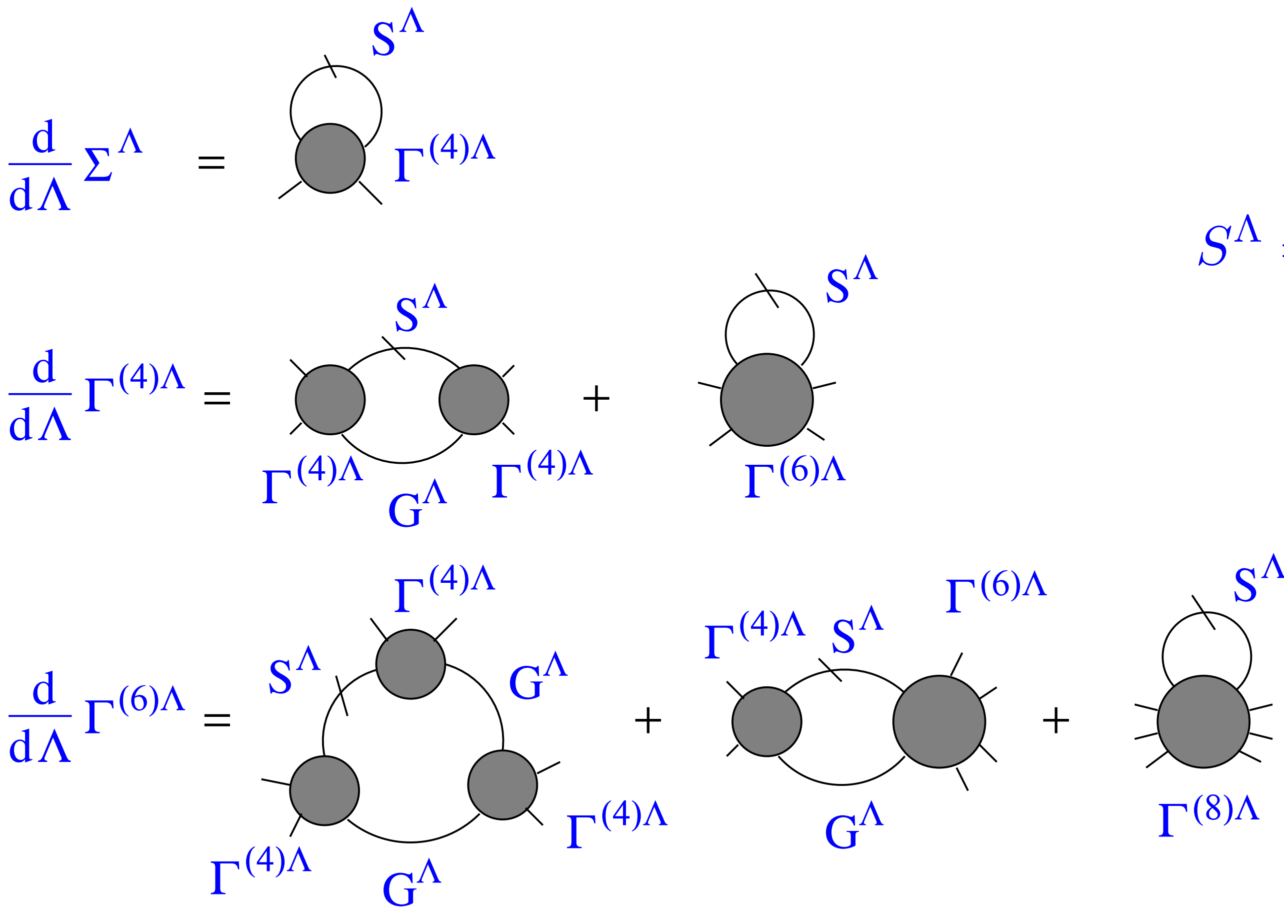
$$\left(\mathbf{\Gamma}^{(2)\Lambda}[\psi,\bar\psi]\right)^{-1} = (1 - \mathbf{G}^\Lambda\tilde{\mathbf{\Sigma}}^\Lambda)^{-1}\,\mathbf{G}^\Lambda = \left[1 + \mathbf{G}^\Lambda\tilde{\mathbf{\Sigma}}^\Lambda + (\mathbf{G}^\Lambda\tilde{\mathbf{\Sigma}}^\Lambda)^2 + \dots\right]\mathbf{G}^\Lambda \;\Rightarrow$$

$$\boxed{\frac{d}{d\Lambda}\Gamma^\Lambda = -\mathrm{tr}\left[\dot{Q}_0^\Lambda G^\Lambda\right] - (\bar\psi, \dot{Q}_0^\Lambda\psi) + \frac{1}{2}\mathrm{tr}\left[\mathbf{S}^\Lambda(\tilde{\mathbf{\Sigma}}^\Lambda + \tilde{\mathbf{\Sigma}}^\Lambda\mathbf{G}^\Lambda\tilde{\mathbf{\Sigma}}^\Lambda + \dots)\right]}$$

where $\mathbf{S}^\Lambda = -\mathbf{G}^\Lambda\dot{\mathbf{Q}}_0^\Lambda\mathbf{G}^\Lambda = \dfrac{d}{d\Lambda}\,\mathbf{G}^\Lambda\big|_{\mathbf{\Sigma}^\Lambda\ \mathrm{fixed}}$ $\quad$ *"single scale propagator"*

Expand $\Gamma^\Lambda[\psi,\bar\psi]$ and $\tilde{\mathbf{\Sigma}}^\Lambda[\psi,\bar\psi]$ in powers of ψ and $\bar\psi$, compare coefficients $\quad\Rightarrow$

Flow equations for self-energy $\Sigma^\Lambda = Q_0^\Lambda - \Gamma^{(2)\Lambda}$, two-particle vertex $\Gamma^{(4)\Lambda}$, and many-particle vertices $\Gamma^{(6)\Lambda}$, $\Gamma^{(8)\Lambda}$, etc.



$$S^\Lambda = \frac{d}{d\Lambda} G^\Lambda \Big|_{\Sigma^\Lambda \text{ fixed}}$$

Hierarchy of 1-loop diagrams; all one-particle irreducible

Initial conditions:

Σ^{Λ_0} = bare single-particle potential (if any)

$\Gamma^{(4)\Lambda_0}$ = antisymmetrized bare two-particle interaction

$\Gamma^{(2m)\Lambda_0} = 0$ for $m \geq 3$

$\Gamma^{\Lambda}[\psi, \bar{\psi}]$ interpolates between regularized bare action $\mathcal{S}^{\Lambda_0}[\psi, \bar{\psi}]$ for $\Lambda = \Lambda_0$ and generating functional for vertex functions $\Gamma[\psi, \bar{\psi}]$ for $\Lambda = 0$.

2.3. Truncations

Infinite hierarchy of flow equations usually unsolvable.

Two types of approximation:

- Truncation of hierarchy at finite order
- Simplified parametrization of effective interactions

Truncations can be justified for weak coupling or small phase space.

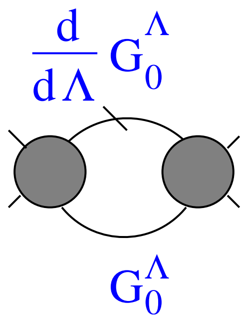
For bosonic fields (e.g. order parameter fluctuations) non-perturbative truncations, including all powers in fields, are possible.

See, for example, Jakubczyk et al., PRL **103**, 220602 (2009)

Simple truncations:

- Set $\Gamma^{(2m)\Lambda} = 0$ for $m > 2$, neglect self-energy feedback in flow of $\Gamma^{\Lambda} \equiv \Gamma^{(4)\Lambda}$:

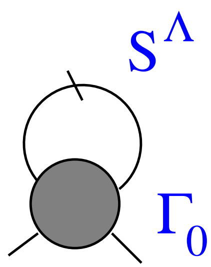
$$\frac{d}{d\Lambda}\Gamma^{\Lambda} =$$



Unbiased stability analysis
at weak coupling;
d-wave superconductivity
in 2D Hubbard model

- Compute flow of self-energy with bare interaction (neglecting flow of Γ^{Λ}):

$$\frac{d}{d\Lambda}\Sigma^{\Lambda} =$$

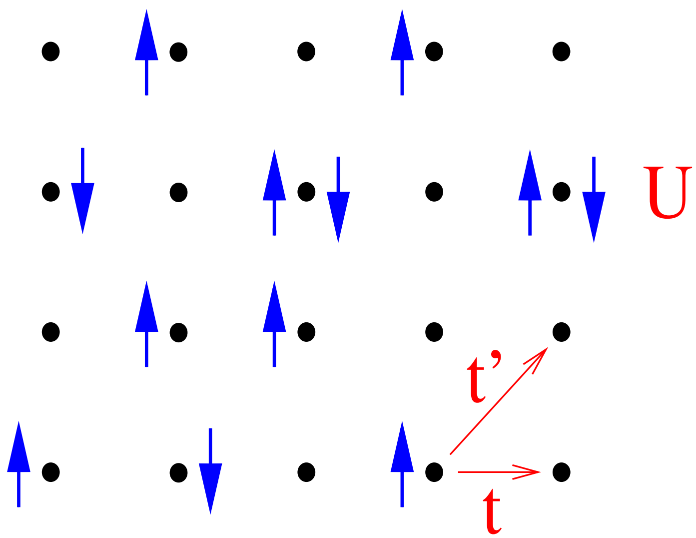


Captures properties
of isolated impurities
in 1D Luttinger liquid

Rev. Mod. Phys. **84**, 299 (2012)

3. Two-dimensional Hubbard model

Effective single-band model for $\mathrm{CuO_2}$-planes in HTSC:
(Anderson '87, Zhang & Rice '88)



Hamiltonian $H = H_{kin} + H_I$

$$H_{kin} = \sum_{\mathbf{i},\mathbf{j}} \sum_{\sigma} t_{\mathbf{ij}}\, c^\dagger_{\mathbf{i}\sigma} c_{\mathbf{j}\sigma} = \sum_{\mathbf{k},\sigma} \epsilon_{\mathbf{k}}\, n_{\mathbf{k}\sigma}$$

$$H_I = U \sum_{\mathbf{j}} n_{\mathbf{j}\uparrow} n_{\mathbf{j}\downarrow}$$

Antiferromagnetism at/near half-filling for sufficiently large U

d-wave superconductivity away from half-filling
(perturbation theory, RG, cluster DMFT, variational MC, some QMC)

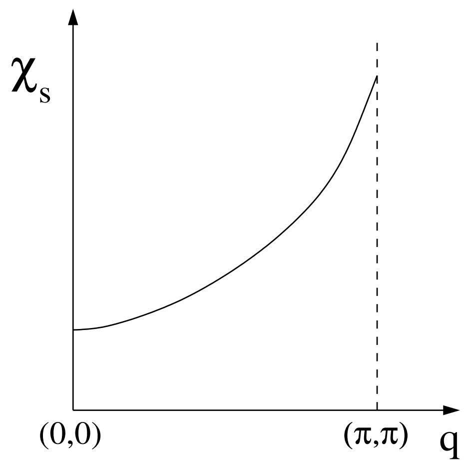
Superconductivity from spin fluctuations:

Miyake, Schmitt-Rink, Varma '86

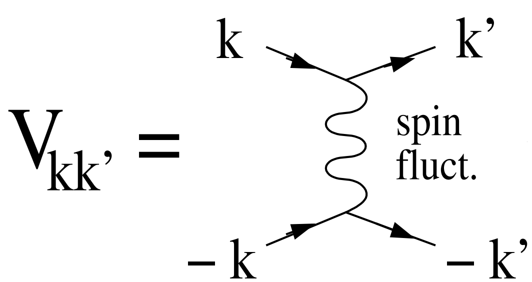
Scalapino, Loh, Hirsch '86

Spin correlation function $\chi_s(\mathbf{q})$ near half-filling:

maximum near (π, π)



Effective BCS interaction from exchange of spin fluctuations

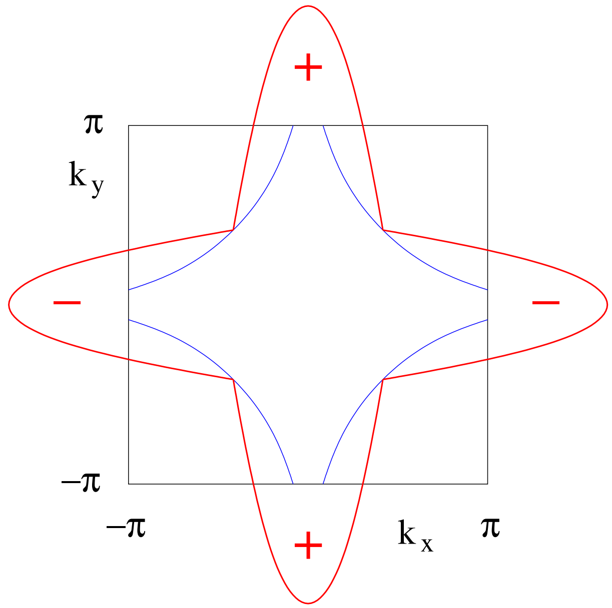


peaked for $\mathbf{k}' - \mathbf{k} = (\pi, \pi)$

$\Rightarrow$ Gap equation

$$\Delta_{\mathbf{k}} = -\sum_{\mathbf{k}'} V_{\mathbf{k}\mathbf{k}'} \frac{\Delta_{\mathbf{k}'}}{2E_{\mathbf{k}'}}$$

has solution with d-wave symmetry



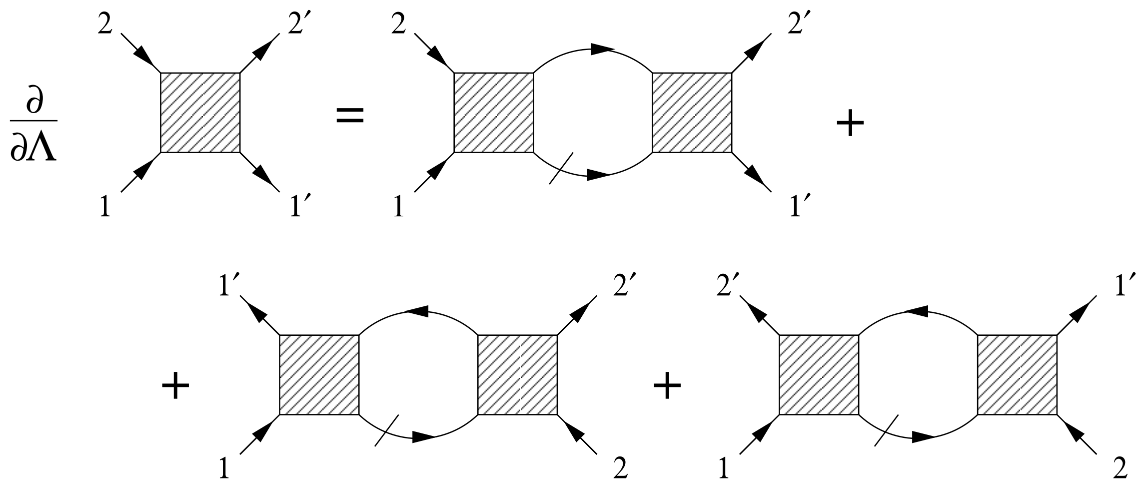
What about other (than AF spin) fluctuations?

Treat all particle-particle and particle-hole channels on equal footing

$\Rightarrow$ Summation of parquet diagrams (hard) or RG

3.1. Stability analysis at weak coupling

Effective 2-particle interaction Γ^Λ at 1-loop level:



bare propagators

G_0^Λ and $\partial_\Lambda G_0^\Lambda$

All channels (particle-particle, particle-hole) captured on equal footing.

Contributions of order $(\Gamma^\Lambda)^3$ neglected.

Explicitly:

$$\begin{aligned}
\frac{\partial}{\partial\Lambda}\Gamma^{\Lambda}(K_1',K_2';K_1,K_2) \quad &= \quad -\frac{1}{\beta V}\sum_{K,K'}\frac{\partial}{\partial\Lambda}\left[G_0^{\Lambda}(K)\,G_0^{\Lambda}(K')\right] \\
&\times \quad \Big[\frac{1}{2}\,\Gamma^{\Lambda}(K_1',K_2';K,K')\,\Gamma^{\Lambda}(K,K';K_1,K_2) \\
&\quad -\Gamma^{\Lambda}(K_1',K';K_1,K)\,\Gamma^{\Lambda}(K,K_2';K',K_2) \\
&\quad +\Gamma^{\Lambda}(K_2',K';K_1,K)\,\Gamma^{\Lambda}(K,K_1';K',K_2)\Big]
\end{aligned}$$

Spin structure: For a spin-rotation invariant system

$$\Gamma^{\Lambda}(K_1',K_2';K_1,K_2) = \Gamma_s^{\Lambda}(k_1',k_2';k_1,k_2)\,S_{\sigma_1',\sigma_2';\sigma_1,\sigma_2} + \Gamma_t^{\Lambda}(k_1',k_2';k_1,k_2)\,T_{\sigma_1',\sigma_2';\sigma_1,\sigma_2}$$

where

$$S_{\sigma_1',\sigma_2';\sigma_1,\sigma_2} = \tfrac{1}{2}\left(\delta_{\sigma_1\sigma_1'}\delta_{\sigma_2\sigma_2'} - \delta_{\sigma_1\sigma_2'}\delta_{\sigma_2\sigma_1'}\right) \qquad \text{singlet}$$

$$T_{\sigma_1',\sigma_2';\sigma_1,\sigma_2} = \tfrac{1}{2}\left(\delta_{\sigma_1\sigma_1'}\delta_{\sigma_2\sigma_2'} + \delta_{\sigma_1\sigma_2'}\delta_{\sigma_2\sigma_1'}\right) \qquad \text{triplet}$$

Carry out spin sum $\Rightarrow$

$$\partial_\Lambda \Gamma^\Lambda_\alpha(k_1', k_2'; k_1, k_2) =$$

$$-\sum_{\substack{i=s,t\\ j=s,t}} \left[C^{\mathrm{PP}}_{\alpha ij}\beta^{\mathrm{PP}}_{ij}(k_1', k_2'; k_1, k_2) + C^{\mathrm{PH}}_{\alpha ij}\beta^{\mathrm{PH}}_{ij}(k_1', k_2'; k_1, k_2) + C^{\mathrm{PH}'}_{\alpha ij}\beta^{\mathrm{PH}'}_{ij}(k_1', k_2'; k_1, k_2)\right]$$

where $C^{\mathrm{PP}}_{\alpha ij}$ etc. are simple coefficients and

$$\beta^{\mathrm{PP}}_{ij}(k_1', k_2'; k_1, k_2) = \frac{1}{2\beta V}\sum_{k,k'} \partial_\Lambda \left[G_0^\Lambda(k)\, G_0^\Lambda(k')\right] \,\Gamma^\Lambda_i(k_1', k_2'; k, k')\,\Gamma^\Lambda_j(k, k'; k_1, k_2)$$

$$\beta^{\mathrm{PH}}_{ij}(k_1', k_2'; k_1, k_2) = -\frac{1}{\beta V}\sum_{k,k'} \partial_\Lambda \left[G_0^\Lambda(k)\, G_0^\Lambda(k')\right] \,\Gamma^\Lambda_i(k_1', k; k_1, k')\,\Gamma^\Lambda_j(k', k_2'; k, k_2)$$

$$\beta^{\mathrm{PH}'}_{ij}(k_1', k_2'; k_1, k_2) = -\beta^{\mathrm{PH}}_{ij}(k_2', k_1'; k_1, k_2)$$

Translation invariance: $\Gamma^\Lambda_\alpha(k_1', k_2'; k_1, k_2) \neq 0$ only for $k_1 + k_2 = k_1' + k_2'$

Parametrization of vertex:

One-loop flow given by non-linear integro-differential equation.

Γ^Λ has three independent momentum and frequency variables.

Brute force discretization doesn't work $\Rightarrow$

Use approximate parametrization of Γ^Λ with a tractable number of variables.

Standard approach:

- Neglect energy dependence: $\Gamma^\Lambda_\alpha(k'_1, k'_2; k_1, k_2) \approx \Gamma^\Lambda_\alpha(\mathbf{k}'_1, \mathbf{k}'_2; \mathbf{k}_1, \mathbf{k}_2)$
- Neglect momentum dependence perpendicular to Fermi surface

  (irrelevant in powercounting)

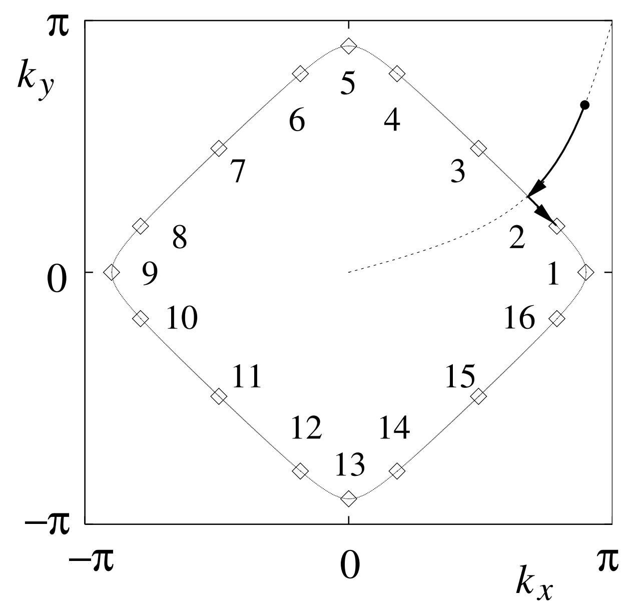
Projection on Fermi surface:

- Neglect momentum dependence perpendicular to Fermi surface
- Keep tangential momentum dependence

$$\Rightarrow \quad \Gamma^{\Lambda}_{\alpha}(\mathbf{k}'_1, \mathbf{k}'_2; \mathbf{k}_1, \mathbf{k}_2) \approx \Gamma^{\Lambda}_{\alpha}(\mathbf{k}'_{F1}, \mathbf{k}_{F1} + \mathbf{k}_{F2} - \mathbf{k}'_{F1}; \mathbf{k}_{F1}, \mathbf{k}_{F2})$$

$\mathbf{k}_{F1}, \ldots =$ projection of $\mathbf{k}_1, \ldots$ on Fermi surface.

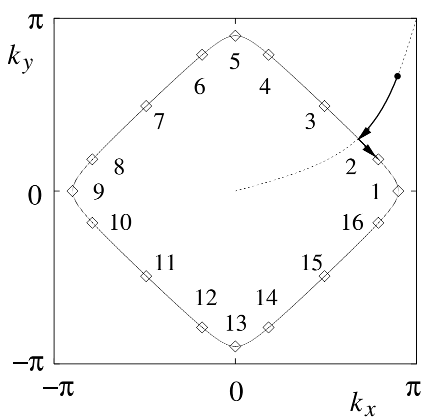
Tangential momentum dependence discretized for numerical solution of flow equations

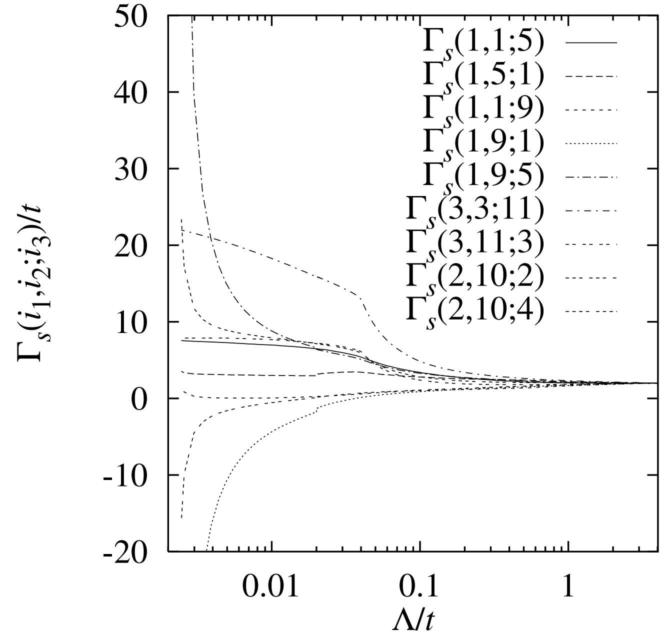


Equivalent to discretization via partition of Brillouin zone in "patches"

1-loop flow:

2-particle interactions





Singlet vertex

$\Gamma_s^\Lambda(k_1', k_2'; k_1, k_2)$

for various choices

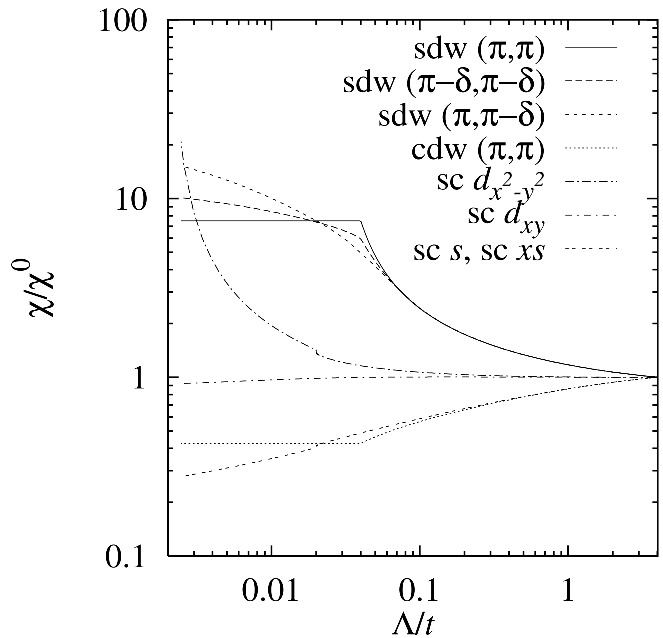
of k_1, k_2, k_1'

Divergence

at critical scale Λ_c

indicates instability

Susceptibilities



$n = 0.984$

$U = t$

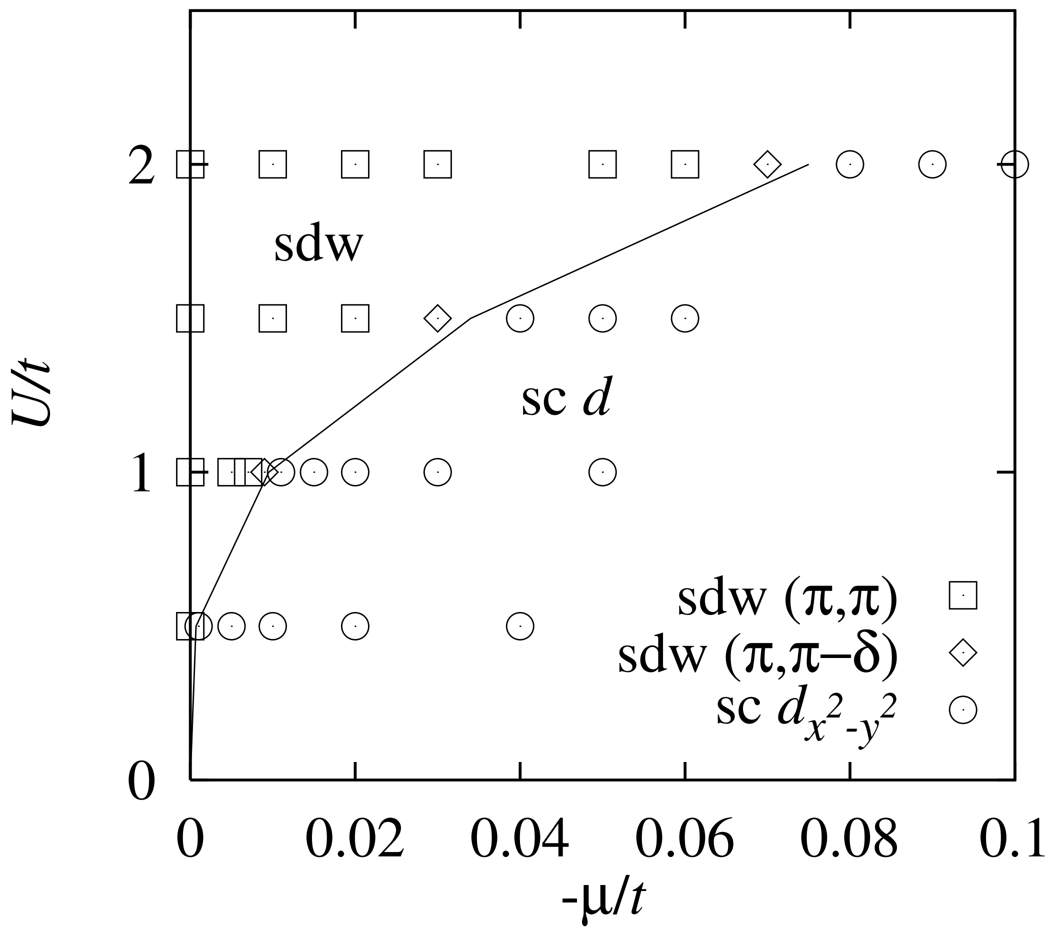
$t' = 0$

Zanchi & Schulz '97-'00

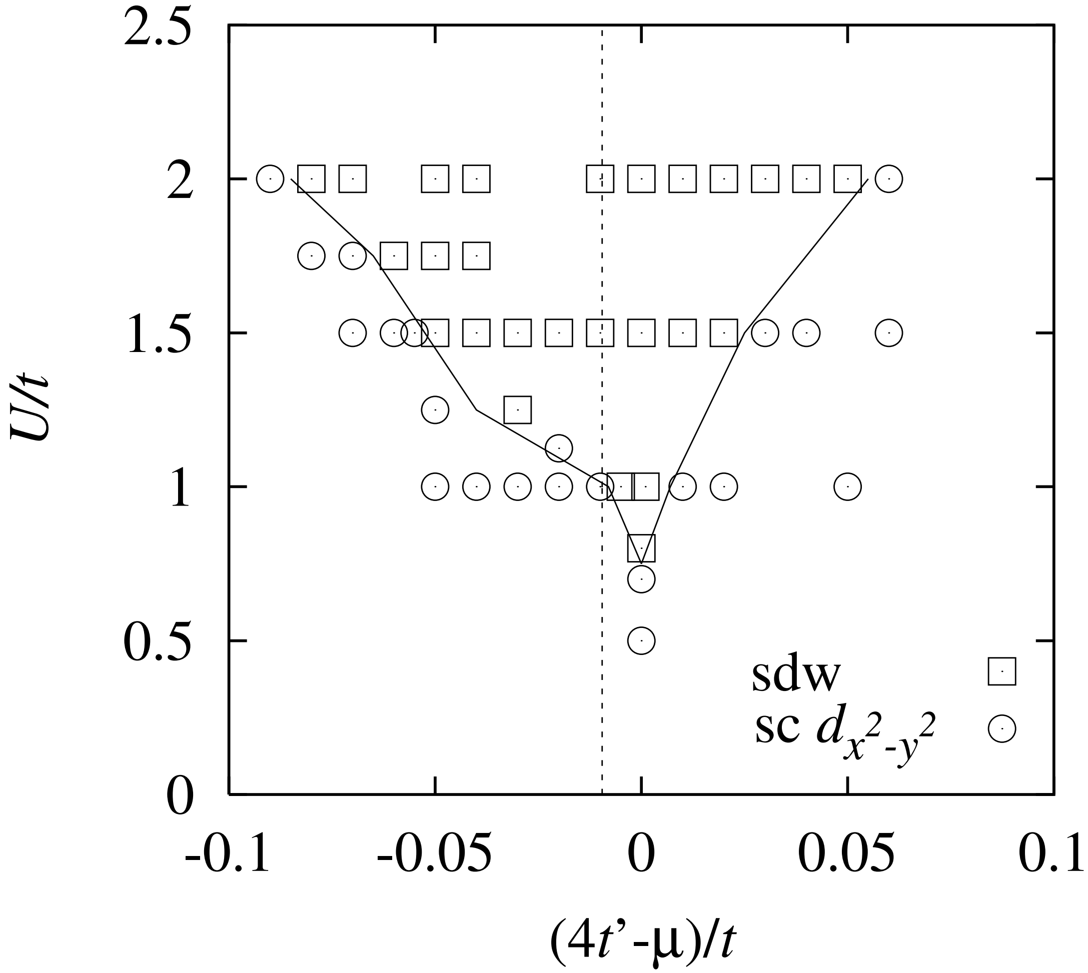
Halboth & wm 2000

Honerkamp et al. 2001

Tentative ground state phase diagram near half-filling, for $t' = 0$:
(from largest susceptibilities)



Tentative ground state phase diagram near half-filling, for $t' = -0.01t$:
(from largest susceptibilities)



Lessons from 1-loop flow (obtained by 2000):

- Strong antiferromagnetic correlations near half-filling
- Antiferromagnetic correlations drive d-wave pairing instability
- Other pairing correlations suppressed
- Conventional charge density waves suppressed
- d-wave charge correlations generated

3.2. Spontaneous symmetry breaking

Divergence of effective interactions at scale Λ_c signals spontaneous symmetry breaking

$\Rightarrow$ Order parameter generated

Routes to symmetry breaking in fRG:

- Fermionic flow with order parameter (Salmhofer, Honerkamp, wm, Lauscher 2004)
- Bosonization (Hubbard-Stratonovich) (Baier, Bick, Wetterich 2004)

Flow into symmetry broken phase complicated, especially in case of two or more order parameters

Cheaper: fRG + mean field theory

Functional RG + mean-field treatment: Wang, Eberlein, wm 2014

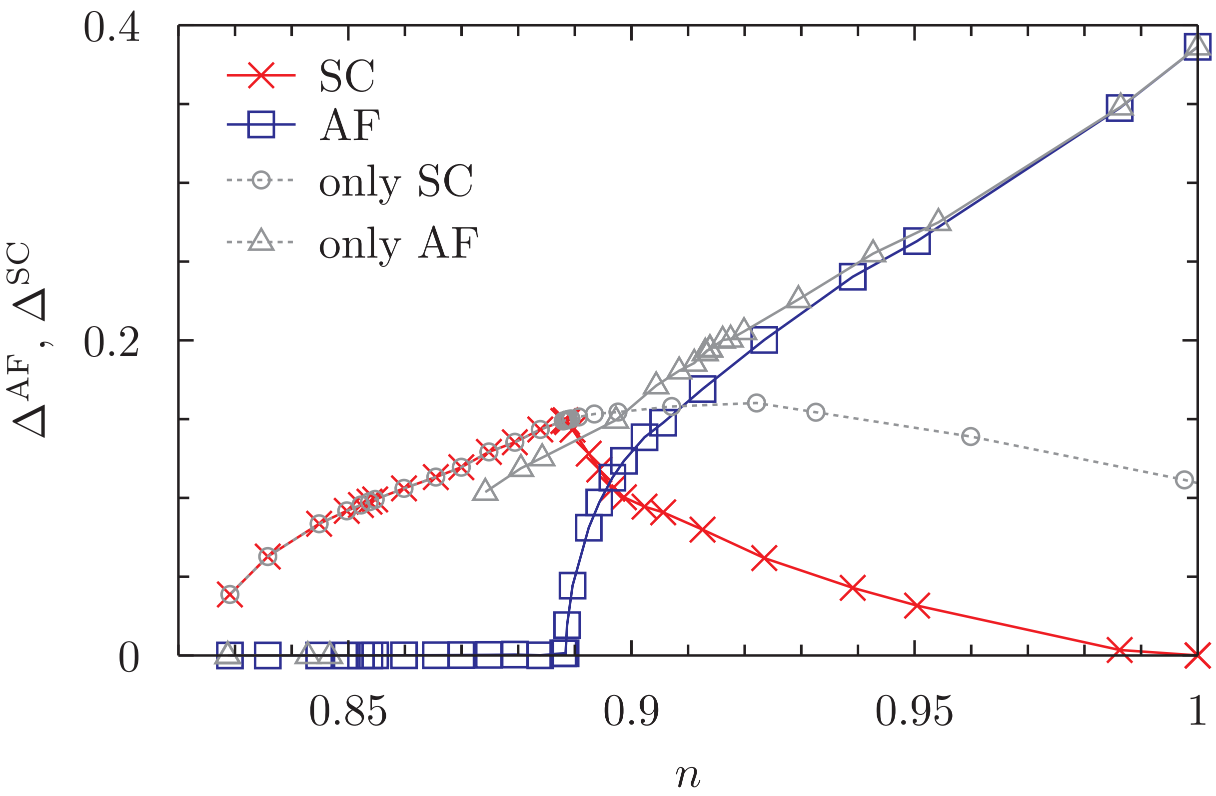
Example for fusion of distinct approximations at high and low energy scales

- Stop fRG flow at scale $\Lambda_{\mathrm{MF}} > \Lambda_c$: effective interaction $\Gamma^{\Lambda_{\mathrm{MF}}}$
- Treat flow for $\Lambda < \Lambda_{\mathrm{MF}}$ in mean-field theory with $\Gamma^{\Lambda_{\mathrm{MF}}}$ as input
  (previous work with Wick ordered fRG: Reiss, Rohe, wm '07)
- $\mathbf{k}$-dependence of order parameter(s) computed, not fixed by ansatz

Application to 2D Hubbard:

Commensurate AF and d-wave SC (coexistence allowed!)

Order parameters (gaps) as a function of density:



$t'/t = -0.15$

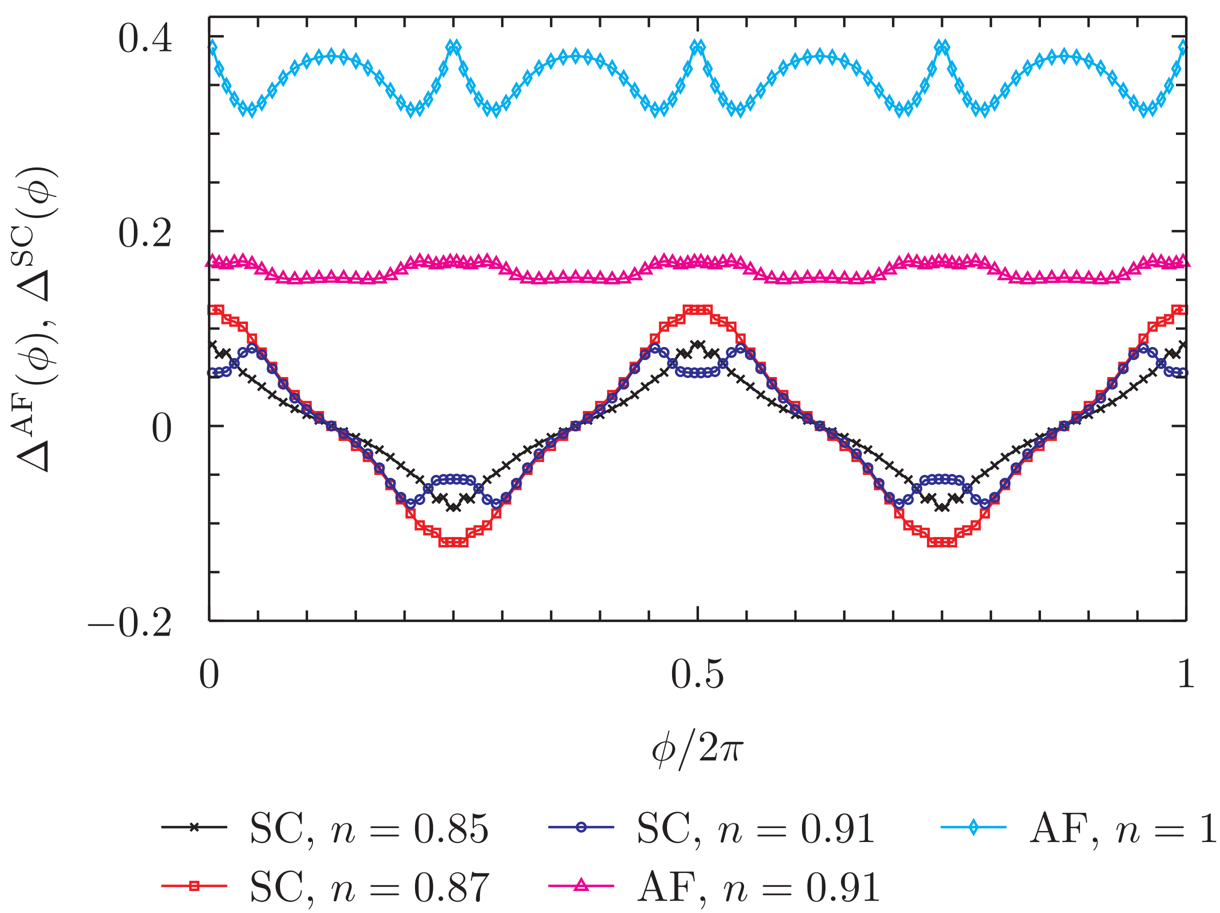
$U/t = 3$

$T = 0$

Wang et al. '14

Coexistence of antiferromagnetism and superconductivity

Momentum dependence of gap functions:



$t'/t = -0.15$

$U/t = 3$

$T = 0$

Wang et al. '14

4. Leap to strong coupling: DMFT as a booster rocket

Truncation of exact flow equation hierarchy justified only for weak interactions

Mott insulator physics in strongly interacting Hubbard model *not* captured by weak coupling expansion

Leap to strong coupling:

Start flow from DMFT ($d = \infty$) !

Taranto, Andergassen, Bauer, Held, Katanin, wm, Rohringer, Toschi, PRL 2014

4.1. Dynamical mean-field theory

4.2. From infinite to finite dimensions

4.3. Application to 2D Hubbard model

4.1. Dynamical mean-field theory

wm & Vollhardt 1989

Georges & Kotliar 1992

DMFT $=$ *local* approximation for self-energy (and other vertex functions):
All vertices in skeleton expansion collapsed to the same lattice site.

Exact in infinite dimensions

Self-energy $\Sigma(k_0) \equiv \Sigma_{\mathbf{jj}}(k_0)$ functional of local propagator $G_{\text{loc}}(k_0) = \int_{\mathbf{k}} G(k_0, \mathbf{k})$

Collapsed skeleton expansion of lattice self-energy same as that for local auxiliary action (↔ single-impurity Anderson model)

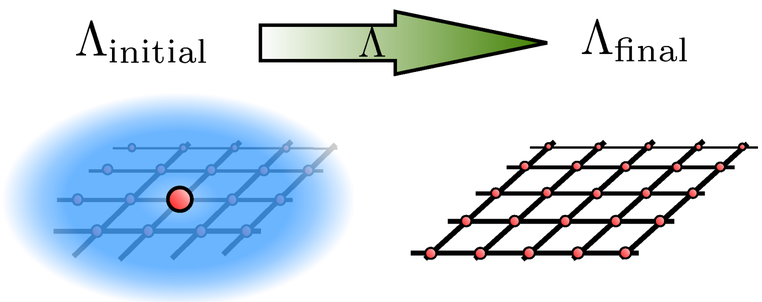
$$\mathcal{S}_{\text{loc}}[\psi, \bar{\psi}] = -\sum_{k_0,\sigma} \bar{\psi}_{k_0,\sigma} \mathcal{G}_0^{-1}(k_0) \psi_{k_0,\sigma} + U \int_0^\beta d\tau\, \bar{\psi}_\uparrow(\tau)\psi_\uparrow(\tau)\bar{\psi}_\downarrow(\tau)\psi_\downarrow(\tau)$$

with Weiss field $\mathcal{G}_0^{-1}(k_0)$

Self-consistency condition: $G_{\text{loc}}^{-1}(k_0) = \mathcal{G}_0^{-1}(k_0) - \Sigma(k_0)$

4.2. From infinite to finite dimensions

Idea: Construct fRG flow that interpolates smoothly between DMFT action and exact action of d-dimensional system.



Wetterich's flow equation holds for *any* modification of quadratic part of action

Simple interpolation: $\left[G_0^\Lambda(k_0,\mathbf{k})\right]^{-1} = \Lambda\, \mathcal{G}_0^{-1}(k_0) + (1-\Lambda) G_0^{-1}(k_0,\mathbf{k})$

Weiss field bare lattice propagator

Initial condition: $\Sigma^{\Lambda_0} = \Sigma_{\mathrm{DMFT}}$ $\Gamma^{(2m)\Lambda_0} = \Gamma^{(2m)}_{\mathrm{DMFT}}$

Alternative interpolation: dimensional flow

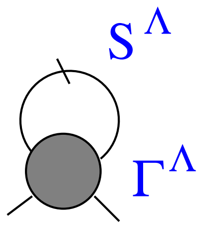
Start from infinite dimensional lattice and smoothly switch off hopping amplitudes in extra dimensions.

4.3. Application to 2D Hubbard model

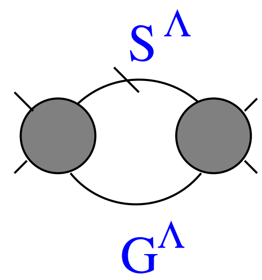
Non-local correlations expected to be weaker than local correlations

$\Rightarrow$ Keep only Σ^Λ and two-particle vertex Γ^Λ in flow equations:

$$\frac{d}{d\Lambda}\Sigma^\Lambda =$$



$$\frac{d}{d\Lambda}\Gamma^\Lambda =$$



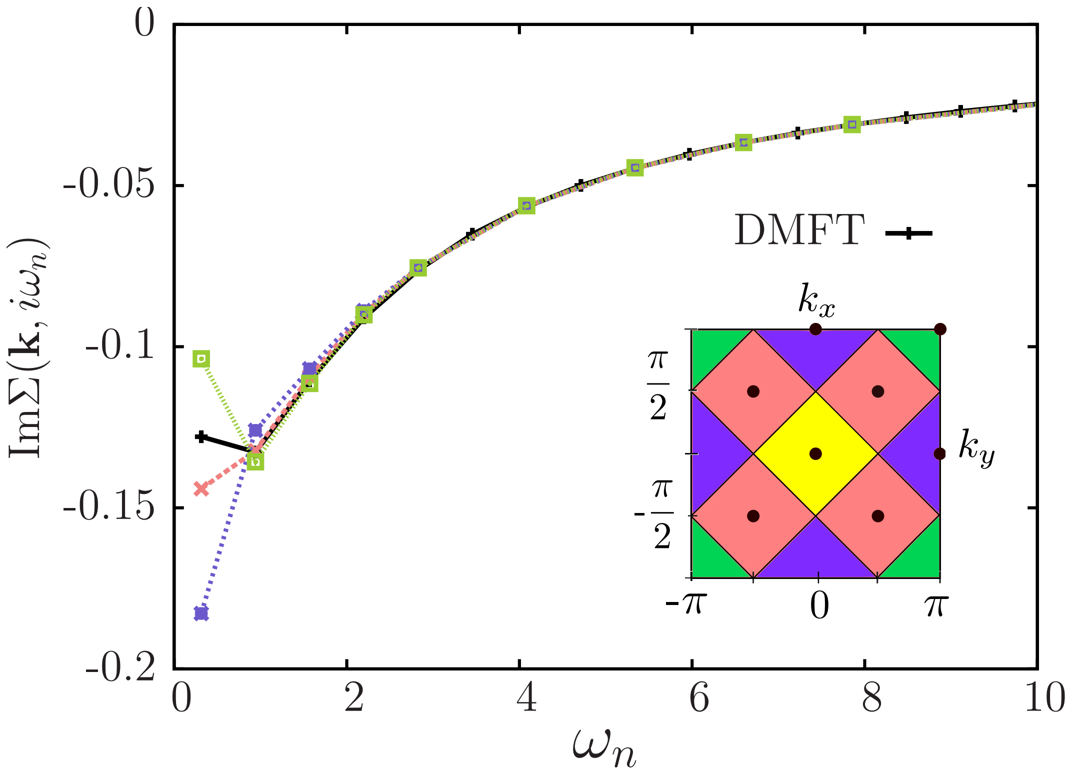
Frequency dependence of vertex approximated by channel decomposition

$$\begin{aligned}\Gamma^\Lambda = U &+ \Gamma^\Lambda_{\mathrm{PP}}(\mathbf{k}_1', \mathbf{k}_2'; \mathbf{k}_1, \mathbf{k}_2; \omega_{\mathrm{PP}}) \\ &+ \Gamma^\Lambda_{\mathrm{PH}}(\mathbf{k}_1', \mathbf{k}_2'; \mathbf{k}_1, \mathbf{k}_2; \omega_{\mathrm{PH}}) + \Gamma^\Lambda_{\mathrm{PH}'}(\mathbf{k}_1', \mathbf{k}_2'; \mathbf{k}_1, \mathbf{k}_2; \omega_{\mathrm{PH}'})\end{aligned}$$

(valid only at weak to moderate U; improvements in progress)

Momentum dependence discretized with few patches

Result for self-energy at half-filling:



pure nearest-neighbor hopping t

$U = 4t$

$T = 0.4t$

Pronounced momentum dependence at low frequency

Saddle points $(\pi, 0)$ and $(0, \pi)$ more "insulating" than other $\mathbf{k}$

$\longrightarrow$ pseudo gap

Conclusion

The fRG framework is a powerful source of new approximations, dealing efficiently with the hierarchy of energy scales in interacting electron systems.

- applicable to microscopic models (not only field theory)
- RG treatment of infrared singularities built in
- consistent fusion of distinct scale-dependent approximations
- applicable to strongly interacting electrons with DMFT as a booster rocket